THE LAWS OF ALGEBRA

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AN ELEMENTARY COURSE IN ALGEBRAIC THEORY

BY

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PREFACE.

It has long been acknowledged that there is great need of an elementary account of the Laws of Algebra suitable for use in the senior classes of schools. The object of this little book is to meet that need: it makes, therefore, no pretence of expounding new ideas, but aims at a clear and simple statement of the rudiments of algebraic theory.

The recognised methods of teaching algebra in schools assume rightly that Formal Algebra and the Theory of Numbers are university subjects and quite beyond the mental range of the school boy. For this reason the various principles involved in algebraic methods are either demonstrated empirically or justified by various illustrations mostly of a geometrical nature.

In dealing with beginners this is inevitable. The danger is that the fundamental principles may remain for ever buried under the illustrations or distorted by irrelevant considerations. Many a schoolboy will—with the full approval of his conscience—prove that $a(b+c) = ab + ac$ from Euclid II. 1, and *vice versa.* Pure algebra is independent of geometry and at some period in the school career this ought to be emphasised by instruction in the elements of algebraic theory.

In order to give the simplest possible account of the Laws of Algebra this book has been based upon the ordinary arithmetical concepts of number, integral and fractional, and no attempt has

been made to deal with imaginary numbers, infinities, limiting values, or irrational indices.

Special care has been given to the treatment of the Laws of Signs. In the first place these are established in the cases where the signs have their ordinary arithmetical meanings of addition and subtraction. The use of the signs in connection with directed numbers is then explained and also the necessity in this connection for treating the Laws of Signs as definitions.

Since the positive and negative numbers of pure algebra obey the Laws of Signs by definition, it follows that we may not use these numbers in geometrical and physical calculations until we have shown that the calculations in question are consistent with these Laws. In other words, the Laws of Signs must be verified independently in the case of every formula in which positive and negative numbers are used. Typical instances of this are given from Mechanics and Coordinate Geometry, and certain general principles are established.

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CONTENTS.

CHAPTER I.

THE FUNDAMENTAL LAWS FOR INTEGERS.

1. Counting.—The first process in calculation is that of **counting.** We have a fixed sequence of "numbers" 1, 2, 3, 4, 5, etc. In counting a group of things we make the first thing correspond to 1, the next to 2, the next to 3, and so on. If the last thing corresponds (say) to 7, we say that the result of the counting is 7.

Now suppose that we have counted a group of boys and a group of coats, and that the result is 7 in each case. We know that the boys correspond to the set of numbers 1, 2, 3, 4, 5, 6, 7; and the coats correspond to the *same* set of numbers. We infer that we can give one coat to each boy and that no coats will be left over: we say that the number of coats is **equal** to the number of boys. If the coats corresponded to the numbers 1, 2, 3, 4, 5, only, we should know that some boys would be left without coats, and we should say that the number of coats was **less** than the number of boys, or the number of boys was **greater** than the number of coats.

We see then that the object of counting different groups of things, that is of comparing them with the fixed sequence 1, 2, 3, 4, . . ., is to determine what would be the result of *comparing the groups in the same way* with one another.

2. Theorem.—*It is possible to change a series of objects from any one order to any other order, provided that it is allowed to interchange two consecutive objects as often as may be required.*

Illustration. To change from the order $a\,b\,c\,d\,e$ to the order $c\,e\,b\,d\,a$.

In each of the following steps the only alteration is the interchange of two consecutive letters :—

$a\,b\,c\,d\,e$, $a\,c\,b\,d\,e$, $c\,a\,b\,d\,e$	(c is in required position).
$c\,a\,b\,e\,d$, $c\,a\,e\,b\,d$, $c\,e\,a\,b\,d$	(e is in required position).
$c\,e\,b\,a\,d$	(b is in required position).
$c\,e\,b\,d\,a$	(d and a are in required positions).

Proof.—In general it is clear that any required object can be moved forward one place at a time till it occupies the first place; then any other required object can be moved forward one place at a time till it occupies the second place; and so on till the required order is obtained.

3. Theorem.—*The order of counting is indifferent;* that is to say, the result of counting a group of objects will be the same in whatever order the objects may be arranged.

Proof.—Suppose the numbers 1, 2, 3, 4, 5, . . . placed in a row in order. Place the first object under the 1, the next under the 2, and so on till all the objects are used up. The last number with an object below it gives the result of the counting.

Now it is evident that the result of the counting is not affected by the interchange of any two consecutive objects: for the objects will still be arranged one by one below the *same* set of numbers

But by § 2 we can arrange the objects in any required order by repeated interchange of consecutive objects.

Hence the result of the counting will be the same whatever may be the order in which the objects are arranged.

4. Order of Operations.—In a logical discussion of algebraic operations it is essential to decide exactly what *order* of operations is indicated by any given algebraic expression.

In this book $a + b$ means To a add b;
$a \times b$* means Multiply a by b;

When a series of numbers is connected by signs of operation *the operations are to be performed in order from left to right*—excepting in two cases indicated below. Thus

$a + b + c$ means To a add b, to the result add c.
$a \times b \div c$ means Multiply a by b, divide the result by c.

Exception 1. When some of the symbols are included in brackets the operations indicated within the brackets are to be performed before those indicated outside the brackets. Thus

$a \div (b \times c)$ means Multiply b by c, divide a by the result.

Exception 2. Multiplications and divisions are performed before additions and subtractions when they occur in the same

* Some writers take $a \times b$ to indicate a times b, *i.e.* multiply b by a, but this is very inconvenient from the logical point of view.

series of operations—unless otherwise indicated by brackets. Thus

$a + b \div c$ means Divide b by c, add the result to a.

5. Addition.—In **addition** we are given the results of counting certain groups of objects separately, and are required to find the result of counting all the objects as one group.

6. The Commutative Law for Addition.—Required to prove that

$$\boldsymbol{a + b = b + a.}$$

Required to count the number of things in two groups P and Q, given that P contains a things and Q contains b things.

$a + b$ indicates that we begin the counting with the things in group P, and complete the counting with the things in group Q.

$b + a$ indicates the opposite order of counting.

But the order of counting is indifferent. [§ 3

Hence the two results will be identical, that is

$$a + b = b + a.$$

7. The Associative Law for Addition.—Required to prove that

$$\boldsymbol{a + (b + c) = a + b + c.}$$

[*Illustration.* $3 + \quad 13 \quad = 3 + 5 + 8.$]

Given three groups P, Q, and R containing a, b, and c objects respectively.

(i) In $(b + c)$ group R is combined with group Q, to form the group (Q, R).

In $a + (b + c)$ the group (Q, R) is combined with group P.

(ii) In $a + b$ group Q is combined with group P, to form the group (P, Q).

In $a + b + c$ the group R is combined with the group (P, Q).

In either (i) or (ii) the final result is the counting of all the objects in P, Q, and R. It follows that the two results will be identical: that is

$$a + (b + c) = a + b + c.$$

8. Multiplication.—In **multiplication** we are given a number of groups each containing the same number of objects and are required to determine the total number of objects.

Thus $a \times b$ (or b times a) indicates the total number of objects in b groups each containing a objects.

9. The Commutative Law for Multiplication.—Required to prove that

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{a}.$$

Proof.—Given b bags each containing a pennies, the total number of pennies is $a \times b$.

Now by § 3 if the pennies are rearranged and again counted the final result will be the same.

From each bag take one penny and put them all in a box.

From each bag take another penny and put them all in another box: and so on.

Since there were b bags there will be b pennies in each box.

Since there were a pennies in each bag, we shall have used a boxes.

We now have a boxes each containing b pennies, and thus the total number of pennies is $b \times a$.

Thus

$$a \times b = b \times a.$$

On the same principle whenever we have ***b*** *groups each containing* ***a*** *objects we can rearrange the objects into* ***a*** *groups each containing* ***b*** *objects.*

Illustration. In Fig. 1 we have 3 rows each containing 5 stars.

Thus the total number of stars is 5×3.

Also we have 5 columns each containing 3 stars.

Thus the total number of stars is 3×5.

Hence $$5 \times 3 = 3 \times 5.$$

*	*	*	*	*
*	*	*	*	*
*	*	*	*	*

Fig. 1.

The above is a good illustration, but not a good proof; for the principle involved is entirely independent of any geometrical arrangement.

Note.—Since $a \times b = b \times a$ it is often unnecessary to distinguish between the two. In the rest of this book we shall use the notation ab to indicate the result of multiplying either letter by the other, but shall retain the notation $a \times b$ or $b \times a$ when the distinction is of logical importance.

10. The Associative Law for Multiplication.—Required to prove that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}.$$

[*Illustration.* $29 \times 15 = 29 \times 3 \times 5.$]

Proof.—Suppose that we have c boxes each containing b bags each containing a pennies.

(i) The number of bags is $(b \times c)$.

But each bag contains a pennies; hence the total number of pennies is $a \times (b \times c)$.

(ii) The number of pennies in each box is $(a \times b)$.

But there are c boxes; hence the total number of pennies is $(a \times b) \times c$.

These two methods must give the same result. Thus

$$a \times (b \times c) = (a \times b) \times c.$$

Again $(a \times b) \times c$ means exactly the same as $a \times b \times c$. For each means "multiply a by b and the result by c." Thus

$$a \times (b \times c) = a \times b \times c.$$

In general if we have c major groups each containing b minor groups each containing a objects, then

(i) the number of minor groups is $(b \times c)$, and the total number of objects is therefore $a \times (b \times c)$.

(ii) the number of objects in each major group is $(a \times b)$, and the total number of objects is therefore $(a \times b) \times c$.

Hence
$$\begin{aligned} a \times (b \times c) &= (a \times b) \times c \\ &= a \times b \times c.* \end{aligned}$$

11. The Distributive Law.—Required to prove that

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}.$$

[*Illustration.* $3 \times 17 = 3 \times 10 + 3 \times 7$.]

Suppose that we have b groups of a objects and also c groups of a objects.

(i) The total number of groups is $(b + c)$.

Since each group contains a objects, the total number of objects is $a \times (b + c)$.

(ii) The number of objects in the b groups is $a \times b$.

The number of objects in the c groups is $a \times c$.

Hence the total number of objects is $a \times b + a \times c$.

These two methods must give the same result. Thus

$$a \times (b + c) = a \times b + a \times c.$$

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* The two last expressions are equal because they indicate exactly the same operations in the same order.

Corollary.—Applying § 9 to each product in the above we obtain

$$(b + c) \times a = b \times a + c \times a.$$

It is quite easy to prove this independently. For a groups each containing $b + c$ objects can be rearranged to make a groups containing b objects and a groups containing c objects.

12. The Fundamental Laws.—We have now proved the Fundamental Laws of Algebra in the case where all the letters represent positive integers. These Laws are

(i) The Commutative Laws for Addition and Multiplication (§§ 6, 9).

(ii) The Associative Laws for Addition and Multiplication (§§ 7, 10).

(iii) The Distributive Laws (§ 11).

The Commutative Laws deal with change of order.

The Associative Laws deal with the insertion or removal of brackets.

The Distributive Laws enable us to use the sum of two numbers as a multiplier or multiplicand.

CHAPTER II.

THE DERIVED LAWS FOR INTEGERS.

13. Subtraction.—The original meaning of the word subtraction is to take away a group of objects from a larger group (or a part from a whole) and to estimate the remainder.* It is, however, evident that the group removed together with the remainder makes up the original group, so that the result of any subtraction can be expressed as an addition. For the purposes of algebraic theory it is necessary to adopt this second method of defining subtraction.

DEFINITION.—*To* **subtract** *one number from another is to find that number which when added to the first makes up the second.*

Thus, by definition,

$$a - b = c,$$

provided that

$$b + c = a,$$

or (using § 6) provided that

$$c + b = a.$$

14. Cancelling Terms.—It is necessary to prove from this definition of subtraction that

$$a - b + b = a \quad \text{..................................(i)}$$

and

$$a + b - b = a \quad \text{..................................(ii)}$$

The first follows from the definition of $a - b$. For this is a number which gives the result a when b is added to it.

The second follows from the definition of $a + b - b$. For this is a number which gives the result $a + b$ when b is added to it, and the number a evidently satisfies this condition.

Thus we have proved that $-b$ and $+b$ cancel out when occurring together in either order.

* That this is the original meaning is evident from the derivation.

15. The Commutative Law for Terms.

THEOREM.—*In a series of terms any change may be made in the order without altering the result.* [Remember that the operations are supposed to be performed in order from left to right.]

By § 2 it will be sufficient to show that any two consecutive terms may be interchanged without altering the result.

There are three cases, according as the two consecutive terms are both positive, one positive and one negative, or both negative.

CASE I.—Suppose that it is required to show that

$$a+b-c-d+e+f-g+h=a+b-c-d+f+e-g+h,$$

where the terms $+e$ and $+f$ are interchanged.

The series of operations indicated before these two terms is the same in both cases. Let the result be R.

Then it is sufficient to show that

$$R+e+f=R+f+e,$$

since if these results are the same the remaining operations to be performed are again identical.

Now
$$\begin{aligned} R+e+f &= R+(e+f) && [\S\ 7 \\ &= R+(f+e) && [\S\ 6 \\ &= R+f+e && [\S\ 7 \end{aligned}$$

CASE II.—Suppose that consecutive terms $+e$ and $-f$ are to be interchanged. As in Case I. it is sufficient to show that

$$R+e-f=R-f+e.$$

Applying the definition of subtraction (§ 13) to the left-hand side of the equation we see that it is only necessary to prove that if f be added to the right-hand side of the equation the result is $R+e$.

Now
$$\begin{aligned} R-f+e+f &= R-f+f+e && [\text{Case I.} \\ &= R+e. && [\S\ 14 \end{aligned}$$

CASE III.—Suppose that consecutive terms $-e$ and $-f$ are to be interchanged. As in Case I. it is sufficient to show that

$$R-e-f=R-f-e.$$

Applying the definition of subtraction (§ 13) to the left side of the equation we see that it is only necessary to prove that if f be added to the right side of the equation the result is $R-e$.

Now
$$\begin{aligned} R-f-e+f &= R-f+f-e && [\text{Case II.} \\ &= R-e. && [\S\ 14 \end{aligned}$$

This completes the proof.

16. Qualifications.—The preceding result must be qualified by the condition that the operations indicated are possible in the order indicated: in other words, the operations indicated must not at any stage involve the subtraction of a larger number from a smaller, for (at present) this has no meaning.

If, however, we interpret such an equation as $+3-7=-4$ to mean that

$$K+3-7=K-4,$$

where K is any number sufficiently large to render all the operations indicated arithmetically possible, then the preceding argument requires no qualification, and any alteration may be made in the order of a series of terms following K.

17. Brackets.—The importance of the order of operations indicated by an algebraical expression, and the effect of the insertion of brackets upon this order has already been discussed in § 4, but it is well to emphasise it here. It is not too much to say that most of the arguments in this book are unintelligible if due regard is not paid to this sequence. Note the following:—

(1) If in a series of terms a group of terms *at the beginning* is included in brackets, this makes *no difference* to the sequence of operations indicated. For example

$$a+b-c-d+e$$

and

$$(a+b-c)-d+e$$

indicate exactly the same operations in the same order.

On the other hand,

$$a+b-c-d+e$$

and

$$a+(b-c)-d+e$$

do *not* indicate the same order of operations—though, as we shall prove later, the final result would be the same in both cases.

(2) Similarly for a series of multipliers and divisors. Thus

$$a\times b\div c\div d\times e$$

and

$$(a\times b\div c)\div d\times e$$

indicate exactly the same operations in the same order.

(3) A bracketed expression may be treated by the same rules as an ordinary number, for it simply represents a number which is the result of the calculations indicated. For example, since

$$a+b=b+a,$$

it follows that

$$(p-q)+(r-s)=(r-s)+(p-q).$$

18. Associative Laws for Addition and Subtraction.—These are the Laws which govern the removal or insertion of brackets in a series of terms, viz.:—

If the bracketed expression stands first or is preceded by a + sign, remove the brackets without change of signs.

If the bracketed expression is preceded by a minus sign, remove the brackets and reverse each sign within them.

The following typical cases will provide a sufficient proof:—

CASE I.—*Where the bracketed expression comes first.**

To prove that

$$(a + b - c) - d + e - f = a + b - c - d + e - f.$$

In this case the removal of the brackets makes no difference in the operations to be performed or in the order in which they are to be performed.

CASE II.—*Where the bracketed expression is preceded by a plus sign.*

To prove that

$$a + b - c + (d - e + f) = a + b - c + d - e + f.$$

Using the Commutative Law

$$\begin{aligned} a + b - c + (d - e + f) &= (d - e + f) + a + b - c \\ &= d - e + f + a + b - c & &\text{[Case I.} \\ &= a + b - c + d - e + f. & &\text{[Comm. Law} \end{aligned}$$

CASE III.—*Where the bracketed expression is preceded by a minus sign.*

To prove that

$$a + b - c - (d + e - f) = a + b - c - d - e + f.$$

By the definition of subtraction it is sufficient to show that, if $(d + e - f)$ is added to the expression on the right of the equation, the result will be $a + b - c$.

Now

$$\begin{aligned} &(a + b - c - d - e + f) + (d + e - f) \\ &= a + b - c - d - e + f + d + e - f & &\text{[Cases I. and II.} \\ &= a + b - c - d + d - e + e + f - f & &\text{[Comm. Law} \\ &= a + b - c, & &\text{[§ 14} \end{aligned}$$

which proves the required result.

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* And is not preceded by a negative sign.

19. Division.—For the purposes of algebraic theory we must define division as the inverse of multiplication, just as we have defined subtraction as the inverse of addition.

DEFINITION.—*To* **divide** *one number by another is to find that number by which the second number must be multiplied to give the first.*

Thus, by definition, $a \div b = c$,

provided that $b \times c = a$,

or (using § 9) provided that $c \times b = a$.

If we take $b \times c = a$ we may say that c is the *number of times that b is contained in a.*

If we take $c \times b = a$ we may say that c is the result of separating a into b equal parts.

Thus the definition given above includes the two other meanings of division used in arithmetic.

20. Cancelling Multipliers and Divisors.—It is necessary to prove from the definition of division that

$$a \div b \times b = a \quad \text{(i)}$$

and

$$a \times b \div b = a \quad \text{(ii)}$$

The first follows from the definition of $a \div b$. For this is a number which gives the result a when multiplied by b.

The second follows from the definition of $a \times b \div b$. For this is a number which gives the result $a \times b$ when multiplied by b, and the number a evidently satisfies this condition.

21. The Commutative Law for Multipliers and Divisors.

THEOREM.—*In a series of multipliers and divisors any change may be made in the order without altering the result.*

By § 2 it will be sufficient to show that any two consecutive operations can be interchanged without altering the result.

There are three cases, according as the two consecutive operations are both multiplications, one multiplication and one division, or both divisions.

CASE I.—Suppose that it is required to prove that

$$a \times b \div c \div d \times e \times f \div g \times h = a \times b \div c \div d \times f \times e \div g \times h,$$

where the operations $\times e$ and $\times f$ are interchanged.

The series of operations indicated before these two operations is the same in both cases. Let the result be R.

Then it is sufficient to show that

$$R \times e \times f = R \times f \times e$$

since if these results are the same the remaining operations are again identical.

Now
$$\begin{aligned} R \times e \times f &= R \times (e \times f) && [\S\ 10 \\ &= R \times (f \times e) && [\S\ 9 \\ &= R \times f \times e. && [\S\ 10 \end{aligned}$$

CASE II.—Arguing as in Case I. it is sufficient to show that

$$R \times e \div f = R \div f \times e.$$

By the definition of division it is only necessary to prove that if the right-hand side of the equation be multiplied by f the result is $R \times e$.

Now
$$\begin{aligned} R \div f \times e \times f &= R \div f \times f \times e && [\text{Case I.} \\ &= R \times e. && [\S\ 20 \end{aligned}$$

CASE III.—Arguing as in Case I., it is sufficient to show that

$$R \div e \div f = R \div f \div e.$$

By the definition of division it is only necessary to prove that if the right-hand side of the equation be multiplied by f the result is $R \div e$.

Now
$$\begin{aligned} R \div f \div e \times f &= R \div f \times f \div e && [\text{Case II.} \\ &= R \div e. && [\S\ 20 \end{aligned}$$

This completes the proof.

22. Parallel Proofs.—If § 21 be compared with § 15 it is evident that the two proofs are identical except that in the second case the signs $\times$ and $\div$ have been substituted for the signs $+$ and $-$ in the first case.

If the theorems in §§ 6, 7 are compared with those in §§ 9, 10, it is at once evident that if any result has been proved from the first set of theorems a corresponding result, in which the signs $\times$ and $\div$ are substituted for the signs $+$ and $-$, can be proved from the second set of theorems.

23. Qualifications.—The result in § 21 must be qualified by the condition that the operations indicated are possible in the orders indicated.

It must be remembered that up to the present we have dealt with integers only, and that the operation of dividing one integer

by another, if we are restricted to exact integral results, more often fails than succeeds.

24. Associative Laws for Multipliers and Divisors.—These Laws correspond exactly to those given in § 18 and the proofs are obtained by substituting the signs $\times$ and $\div$ for the signs $+$ and $-$. For example, we can prove that

CASE I. $(a \times b \div c) \div d \times e \div f = a \times b \div c \div d \times e \div f.$

CASE II. $a \times b \div c \times (d \div e \times f) = a \times b \div c \times d \div e \times f.$

CASE III. $a \times b \div c \div (d \times e \div f) = a \times b \div c \div d \div e \times f.$

25. Extension of Distributive Law.—It is required to prove that

$$a(b - c) = ab - ac \quad \text{.................................} \quad \text{(i)}$$

$$a(b \pm c \pm d \pm \ldots) = ab \pm ac \pm ad \pm \ldots \quad \text{...............} \quad \text{(ii)}$$

As mentioned in § 9, $a(b - c)$ may be interpreted as either $a \times (b - c)$ or $(b - c) \times a$.

Proof.—(i) From the definition of subtraction it is only necessary to prove that if ac be added to the expression on the left of the equation the result is ab.

Now
$$\begin{aligned} a(b - c) + ac &= a \times (b - c) + a \times c \\ &= a \times \{(b - c) + c\} && [\S\ 11 \\ &= a \times \{b - c + c\} && [\S\ 18 \\ &= a \times b. && [\S\ 14 \end{aligned}$$

(ii) It will be sufficient to show that

$$a(b - c + d - e) = ab - ac + ad - ae.$$

Now
$$\begin{aligned} a(b - c + d - e) &= a\{[b - c + d] - e\} && [\S\ 18 \\ &= a[b - c + d] - ae && [\text{Case (i)} \\ &= a[(b - c) + d] - ae && [\S\ 18 \\ &= a(b - c) + ad - ae && [\S\ 11 \\ &= ab - ac + ad - ae && [\text{Case (i)} \end{aligned}$$

26. Distributive Law for Divisors.—It is required to prove that

$$(b \pm c) \div a = (b \div a) \pm (c \div a).$$

By the definition of division it is sufficient to show that the right-hand side of the equation when multiplied by a gives $(b \pm c)$.

By § 11

$$\begin{aligned} \{(b \div a) \pm (c \div a)\} \times a &= (b \div a) \times a \pm (c \div a) \times a \\ &= b \div a \times a \pm c \div a \times a && [\S\ 24 \\ &= b \pm c. && [\S\ 20 \end{aligned}$$

COROLLARY, $(b \pm c \pm d \pm \ldots) \div a = b \div a \pm c \div a \pm d \div a \pm \ldots$

27. Rule of Signs in Multiplication.—The process of Long Multiplication is based on the extension of the Distributive Law, and the Rule of Signs used in this process is merely a convenient mechanical device for determining the signs of the terms in the product.

Consider the example

$$(a + b - c) \times (d - e + f).$$

By § 25 this is equal to

$$(a + b - c) \times d - (a + b - c) \times e + (a + b - c) \times f.$$

By § 4 we must perform the multiplications before the addition and subtraction. Thus applying § 25 we obtain

$$(ad + bd - cd) - (ae + be - ce) + (af + bf - cf) \qquad \text{(i)}$$

By § 18 this gives

$$ad + bd - cd - ae - be + ce + af + bf - cf.$$

Now it is evident that we can obtain the different terms in the final result by multiplying each term in the multiplicand by each term in the multiplier in succession, provided that we can get a working rule for determining the signs of these terms in the product.

For this purpose examine the line marked (i).

The terms *in each bracket* have the same signs as the corresponding terms in the multiplicand $(a + b - c)$.

In the case of the first and third brackets (which correspond to the positive terms in the multiplier) these signs are unaltered when the brackets are removed.

In the case of the second bracket (which corresponds to the negative term in the multiplier) these signs are all altered when the brackets are removed.

Thus the rule for determining the signs of the terms in the product is

RULE.—*Retain the signs in the multiplicand when using a positive term in the multiplier, and alter the signs in the multiplicand when using a negative term in the multiplier.*

All the possible cases of this rule are

$$(+) \times (+) = (+) \qquad (+) \times (-) = (-)$$
$$(-) \times (+) = (-) \qquad (-) \times (-) = (+)$$

These four cases are concisely summarised in the usual rule

Like signs give $+$, *unlike signs give* $-$.

28. The Derived Laws.—The theorems proved in this Chapter form the basis of the various methods of calculation used in arithmetic and algebra.

They are all derived from the fundamental laws mentioned in § 12, together with the definitions of subtraction and division given in §§ 13, 19.

It must be particularly noted that whereas Chapter I. deals with integers only, the proofs of the Derived Laws given in Chapter II. do *not* assume that the letters represent integers, but only that they represent numbers which obey the Fundamental Laws.

It follows that in order to prove that any kind of numbers (*e.g.* fractional numbers or surd numbers) obey the Derived Laws it will only be necessary to prove that they obey the Fundamental Laws. If they do this Chapter shows that they obey the Derived Laws.

The value of this method of treatment lies in the fact that it is quite easy to prove the Fundamental Laws for the different kinds of numbers.

29. Methods of Calculation.—At this stage it will be a valuable exercise for the student to use the Laws already proved to justify the various rules used in algebraic and arithmetical calculations, a list of which is here given.

Exercises.

Prove the following rules in algebra or arithmetic, assuming the Laws of Algebra for all kinds of numbers :—

1. Addition and subtraction of like terms, *e.g.*

$$7a + 4a = 11a, \qquad 7a - 4a = 3a, \qquad [\S\S\ 11,\ 25$$

$$K - 7a + 4a = K - 3a^*. \qquad [\S\S\ 11,\ 18,\ 14$$

2. Product of monomials, *e.g.*

$$7ax \times 3by = 21abxy. \qquad [\S\S\ 24,\ 21$$

NOTE.—$7ax \times 3by = (7 \times a \times x) \times (3 \times b \times y)$.

3. Division of monomials, *e.g.*

$$24axy \div 6x = 4ay. \qquad [\S\S\ 24,\ 21,\ 20$$

NOTE.—$24axy \div 6x = (24 \times a \times x \times y) \div (6 \times x)$.

4. Multiplication and division of powers, *e.g.*

$$x^4 \times x^3 = x^7, \qquad x^5 \div x^2 = x^3. \qquad [\S\S\ 24\ ;\ 24,\ 21,\ 20$$

* Write $-(4 + 3)a$ for $-7a$.

5. Addition and subtraction of compound expressions, *e.g.*

$$\begin{array}{r} 3x + 2y - 7 \\ 4x - 9y \\ 3y + 5 \\ \hline \end{array} \qquad \begin{array}{r} 5x - 2y - 9 \\ 2x + 3y - 4 \\ \hline \end{array}$$

[§§ 18, 15, and Ex. 1

NOTE.—These are equivalent to

$$(3x + 2y - 7) + (4x - 9y) + (3y + 5)$$

and $(5x - 2y - 9) - (2x + 3y - 4)$ respectively.

6. Multiplication of compound expressions and the Rule of Signs. See § 27.

7. Division of compound expressions and the Rule of Signs. [A device for reversing the multiplication process.]

8. To evaluate a series of positive and negative terms, subtract the sum of the negative terms from the sum of the positive terms.
[§§ 15, 18, and Ex. 1

9. In a series of positive and negative terms to remove any positive term is equivalent to subtracting that number from the result, and to remove any negative term is equivalent to adding that number to the result. [§§ 15, 14

10. In a series of multipliers and divisors to remove any multiplier is equivalent to dividing the result by that number, and to remove any divisor is equivalent to multiplying the result by that number.
[§§ 21, 20

11. In a series of positive and negative terms to add a number to one of the positive terms or to subtract a number from one of the negative terms is equivalent to adding that number to the result, and vice versa. [§§ 15, 18

12. In a series of multipliers and divisors to multiply one of the multipliers by a number or to divide one of the divisors by a number is equivalent to multiplying the final result by that number, and vice versa. [§§ 21, 24

13. Justify the process of cancelling by division, *e.g.* show that

$$\frac{3 \times 35 \times 11}{2 \times 15 \times 13} = \frac{3 \times 7 \times 11}{2 \times 3 \times 13}.$$

[§§ 24, 21, 20

14. Which of the fundamental laws are used in the usual arithmetical method of evaluating?—

(i) $2345 + 678$; [§§ 18, 15, 11

(ii) 2345×678. [§§ 25, 21, 11

30. The Laws of Factors.—The Laws of Factors are known empirically from arithmetic. The following formal proofs will furnish a sufficient discussion of the principles involved. The letters in all cases represent integers.

THEOREM I.—*If one number is a factor of another it is a factor of any multiple of that other.*

Given that a is a factor of b,

Required to prove that a is a factor of $b \times m$.

Proof. Since a is a factor of b,

$$b = a \times k \text{ where } k \text{ is integral.}$$

$$\therefore\ b \times m = (a \times k) \times m = a \times (k \times m), \qquad [\S\ 10$$

i.e. a is a factor of $b \times m$.

THEOREM II.—*If one number is a common factor of two other numbers it is a factor of the sum or difference of any multiples of those numbers.*

Given that a is a common factor of b and c,

Required to prove that a is a factor of $mb \pm nc$.

Proof.—Since a is a common factor of b and c,

$$\therefore\ b = ka \text{ and } c = la, \text{ where } k \text{ and } l \text{ are integers,}$$

$$\therefore\ mb \pm nc = mka \pm nla = a\,(mk \pm nl). \qquad [\textit{Dist. Law}$$

That is a is a factor of $mb \pm nc$.

THEOREM III.—*If C is the remainder when A is divided by B, then H.C.F. of A and B = H.C.F. of B and C.*

Suppose the quotient is q.

Then $A = qB + C$ .... (i

whence $C = A - qB$ ....(ii)

From (i) any common factor of B and C is a factor of $qB + C$, that is of A. Hence *any common factor of B and C is a common factor of A and B.*

From (ii) any common factor of A and B is a factor of $A - qB$, that is of C. Hence *any common factor of A and B is a common factor of B and C.*

It follows that the common factors of A and B are identical with those of B and C, and hence that

H.C.F. of A and B = H.C.F. of B and C.

THEOREM IV.—*To prove the Rule for finding the H.C.F. of two numbers by repeated division.*

The process is represented as follows:—

$$
\begin{array}{llll}
B\,)\,A\,(\,p & & & \\
\underline{pB} & & & \\
\quad C\,)\,B\,(\,q & & & \\
\quad\underline{qC} & & & \\
\qquad D\,)\,C\,(\,r & & & \\
\qquad\ \ \cdots & & & \\
\qquad\qquad \cdots & & &
\end{array}
\qquad\qquad
\begin{array}{llll}
\cdots & & & \\
L\,)\,K\,(\,x & & & \\
\underline{xL} & & & \\
\quad M\,)\,L\,(\,y & & & \\
\quad\underline{yM} & & & \\
\qquad N\,)\,M\,(\,z & & & \\
\qquad\quad\underline{zN} & & &
\end{array}
$$

By repeated application of Theorem IV.,

$$
\begin{aligned}
\text{H.C.F. of } A \text{ and } B = \text{H.C.F. of } B \text{ and } C &= \text{H.C.F. of } C \text{ and } D \\
= \ldots\ldots\ldots\ldots\ldots\ldots &= \text{H.C.F. of } K \text{ and } L \\
= \text{H.C.F. of } L \text{ and } M &= \text{H.C.F. of } M \text{ and } N.
\end{aligned}
$$

But N divides exactly into M, $\therefore$ H.C.F. of M and $N = N$.

Hence H.C.F. of A and $B = N$.

THEOREM V.—*If B is a factor of AZ and is prime to A, then B is a factor of Z.*

Since B is prime to A, the H.C.F. of A and B is 1. Thus if we perform the operation of finding the H.C.F. of A and B as represented in Theorem IV., then the last divisor N will be 1.

Now write down the equations corresponding to the successive subtractions in this H.C.F. process. This gives the series of equations in the left-hand column below. Multiplying each of these equations by Z we obtain the equations in the right-hand column.

$$
\begin{array}{l|l}
C = A - pB & CZ = AZ - pBZ \\
D = B - qC & DZ = BZ - qCZ \\
E = C - rD & EZ = CZ - rDZ \\
\ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\
\ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\
M = K - xL & MZ = KZ - xLZ \\
N = L - yM & NZ = LZ - yMZ
\end{array}
$$

Next consider the equations in the right-hand column.

In the first equation, since B is a factor of AZ (hypothesis) and also of BZ,

therefore, by Theorem II., B is a factor of CZ.

In the second equation, since B is a factor of BZ and also of CZ (proved),

therefore, by Theorem II., B is a factor of DZ.

Continuing the argument in this manner we can prove successively that B is a factor of $EZ, \ldots, MZ, NZ$.

But $N = 1$. Hence B is a factor of Z.

THEOREM VI.—*It is not possible to resolve a given number into two different sets of prime factors.*

If possible suppose that a given number can be resolved into two different sets of prime factors, say $abcdef$ and $abcghkl$. Then

$$abcdef = abcghkl,$$

and hence d is a factor of $abcghkl$.

Since d is prime to a and is a factor of $a \times bcghkl$, therefore, by Theorem V., d is a factor of $bcghkl$.

Since d is prime to b and is a factor of $b \times cghkl$, therefore, by Theorem V., d is a factor of $cghkl$.

Proceeding in this way we prove ultimately that d is a factor of l, *which is not the case*, since d is prime to l.

Hence it is not possible to resolve a given number into two different sets of prime factors.

COROLLARY.—By the same line of argument it can be shown that if A is a factor of B and both A and B are resolved into their prime factors, then all the prime factors of A occur among the prime factors of B.

THEOREM VII.—*The L.C.M. of any two numbers is equal to their product divided by their H.C.F.*

Let f be the H.C.F. of the numbers A and B. Then

$$A = af, \; B = bf,$$

where a and b are prime to one another. For if they have a common factor, f is not the H.C.F. of a and b.

To find the L.C.M. of A and B we must find the smallest multiple of A which is divisible by B. That is

$$\text{L.C.M. of } A \text{ and } B = mA,$$

where m is the smallest integer, which makes mA divisible by B.

Now
$$\begin{aligned} mA \div B &= m \times (af) \div (bf) \\ &= m \times a \times f \div b \div f && [\S\,24 \\ &= m \times a \div b \times f \div f && [\S\,21 \\ &= m \times a \div b && [\S\,20 \end{aligned}$$

Thus b must be a factor of ma.

But b is prime to a; hence b must be a factor of m. [Theor. V.

Thus the lowest possible value of m is b.

Hence L.C.M. of A and $B = bA = B \div f \times A = AB \div f$.

CHAPTER III.

FRACTIONS.

31. Ratios and Fractions.—Most calculations are concerned not with groups of indivisible objects, but with magnitudes—such as lengths, areas, volumes, and intervals of time—which we regard as capable of being divided into smaller and smaller parts to any required extent.

Two such magnitudes can be compared with one another by measuring each in terms of some standard unit. The *fundamental* method of comparison, however, is the method of **ratios.** In fact the measure of a magnitude is merely its ratio to the unit.

Suppose that A and B are two magnitudes of the same kind: if we say that the ratio of A to B is 3 to 5, we mean that if A is divided into 3 equal parts then 5 such parts will make up B. In other words, the ratio of A to B tells us by what *operations* B can be obtained from A, the operations being of two kinds, viz. *partition* (dividing A into equal parts) and *repetition* (combining a certain number of such parts to make up B).

Exactly the same relation between A and B can be expressed by **fractions** in either of the forms

$$A = \tfrac{3}{5}B \text{ or } B = \tfrac{5}{3}A.$$

Fractional numbers enable us to measure a magnitude in terms of a unit which is not contained an exact number of times in the magnitude.

Definition.—*To form the fraction* $\dfrac{p}{q}$, *divide the unit into* q *equal parts, and take* p *of them to constitute the fraction. If* $p > q$ *more than one unit must be so divided.*

It follows from this definition $\dfrac{p}{1} = p$.

Note that this definition of a fraction assumes that p and q are both integers. In §§ 32-38 all the letters represent integers.

32. Reduction, Addition, and Subtraction.

THEOREM I.—*To add two fractions which have the same denominator, retain the denominator, and add the numerators.*

Required to prove that $\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r}$.

This follows at once from the definition of a fraction.

For $\frac{p}{r}$ represents p fragments of the unit, and $\frac{q}{r}$ represents q other fragments of the same magnitude. Combining the two we get $(p+q)$ fragments.

Reduction.—The value of a fraction is not altered by multiplying or dividing both numerator and denominator by the same number.

As a numerical instance consider

$$\frac{2}{3}=\frac{5.2}{5.3}=\frac{10}{15}.$$

This is easy to illustrate geometrically.

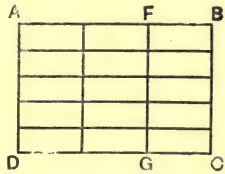

Fig. 2.

Thus in Fig. 2 the vertical lines divide the rectangle into 3 equal parts; so that

rect. $AFGD = \frac{2}{3}$ rect. $ABCD$.

But the vertical and horizontal lines divide the rectangle into 15 equal parts; so that rect. $AFGD = \frac{10}{15}$ rect. $ABCD$.

Hence $\frac{2}{3}=\frac{10}{15}$.

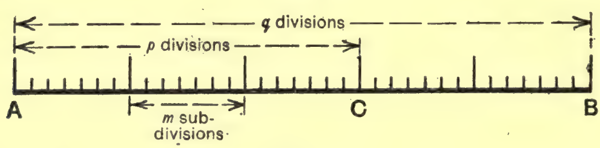

Fig. 3.

THEOREM II.—*The value of a fraction is not altered if the numerator and denominator are both multiplied or both divided by the same number.*

Required to prove that $\frac{mp}{mq}=\frac{p}{q}$.

Proof.—It is convenient to represent the unit by a straight line, but the argument is the same whatever kind of magnitude is under consideration.

Suppose that the unit AB (Fig. 3) is divided (by the large division marks) into q equal divisions. Then the portion AC, which contains p of these equal divisions, constitutes the fraction $\frac{p}{q}$.

[If $p > q$ more than 1 unit must be so divided.]

Now suppose that each of these divisions is divided (by the small division marks) into m equal sub-divisions.

The number of sub-divisions in the unit AB is mq, and the number of sub-divisions in the portion AC is mp.

Thus the portion AC constitutes the fraction $\frac{mp}{mq}$.

Hence $$\frac{mp}{mq} = \frac{p}{q}.$$

Theorem III.—*To prove that* $\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}$.

The value of a fraction is not altered by multiplying numerator and denominator by the same number. [*Theorem II.*

Hence $$\frac{p}{q} = \frac{ps}{qs}, \quad \text{and} \quad \frac{r}{s} = \frac{qr}{qs}.$$

$$\therefore \frac{p}{q} + \frac{r}{s} = \frac{ps}{qs} + \frac{qr}{qs} = \frac{ps + qr}{qs}. \qquad [\textit{Theorem I.}$$

Theorem IV.—To prove that

$$\frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs}.$$

$$\frac{r}{s} + \frac{ps - qr}{qs} = \frac{qr}{qs} + \frac{ps - qr}{qs} \qquad [\textit{Theorem II.}$$

$$= \frac{qr + (ps - qr)}{qs} \qquad [\textit{Theorem I.}$$

$$= \frac{qr + ps - qr}{qs} \qquad [\S\ 18$$

$$= \frac{ps}{qs} \qquad [\S\S\ 15,\ 14$$

$$= \frac{p}{q}. \qquad [\textit{Theorem II.}$$

This proves the result (see § 19).

33. On the Meaning of Multiplication.—Whether a be an integer or a fraction the original definition of multiplication gives that

$$a \times 3 \text{ means } a + a + a,$$

and similarly for a multiplication by any integer.

Evidently the meaning of the word multiplication must be extended if it is to cover the case of fractional multipliers.

We may get any number, integral or fractional, from unity by partition or repetition, or both.

We get 3 from unity by repetition, thus—

$$3 = 1 + 1 + 1.$$

We get $\frac{3}{5}$ from unity by partition (breaking it up into equal parts) and repetition, thus—

Break unity up into 5 equal parts, calling each $\frac{1}{5}$. Then

$$\tfrac{3}{5} = \tfrac{1}{5} + \tfrac{1}{5} + \tfrac{1}{5}.$$

DEFINITION.—*To multiply **a** by **b** is to perform those operations on **a** (operations of partition or repetition or both) which must be performed on unity to obtain **b**.*

34. Illustration.—In order to show that this definition correctly describes the process of multiplying either by an integer or by a fraction it will be sufficient to consider some formula involving multiplication, such as

$$\textit{distance travelled} = \textit{velocity} \times \textit{time},$$

and to work through one or two cases where this formula applies *from first principles.*

Ex. *A steamer travels at the rate of* 8 *miles an hour. How far does it go* (i) *in* 3 *hours,* (ii) *in* $\frac{3}{4}$ *hour?*

(i) In each hour 8 miles is covered.

$$3 \text{ hours} = 1 \text{ hour} + 1 \text{ hour} + 1 \text{ hour}.$$

$$\therefore \text{distance in 3 hours} = 8 \text{ miles} + 8 \text{ miles} + 8 \text{ miles}$$
$$= 24 \text{ miles}.$$

(ii) Divide the hour into 4 equal parts (quarters of an hour) and the 8 miles into 4 equal distances (each 2 miles).

In each quarter of an hour 2 miles is covered.

$$\tfrac{3}{4} \text{ hour} = \tfrac{1}{4} \text{ hour} + \tfrac{1}{4} \text{ hour} + \tfrac{1}{4} \text{ hour}.$$

$$\therefore \text{distance in } \tfrac{3}{4} \text{ hour} = 2 \text{ miles} + 2 \text{ miles} + 2 \text{ miles}$$
$$= 6 \text{ miles}.$$

In the formula both these cases are described as multiplication; also it is clear that in each calculation we have performed the same operations on the 8 miles as must be performed on the unit hour to obtain the given time.

35. Area of a Rectangle.—Again, let us consider the formula for the area of a rectangle, viz.

$$area = length \times breadth.$$

In each of the accompanying figures it is required to find the area of the rectangle $ABCD$, whose length AB is l units and whose breadth AD is b units.

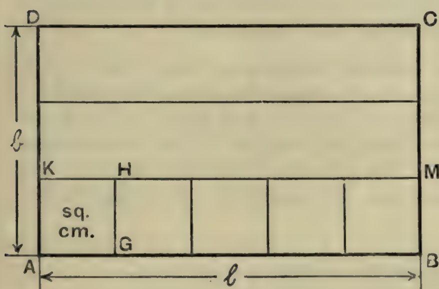

Fig. 4.

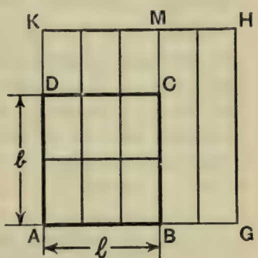

Fig. 5.

In Fig. 4 the unit of length is the centimetre, the unit of area is the square centimetre ($AGHK$), and l and b are integral. In Fig. 5 the unit of length is the inch, the unit of area is the square inch ($AGHK$) and l and b are fractional.

Proof.—In either figure consider first the rectangle $ABMK$ whose length is l units and breadth 1 unit.

It is evident geometrically that this area $ABMK$ is derived from the unit area $AGHK$ by the same operations of partition and repetition as the line AB (l units) is derived from the line AG (unit length).

Thus we may write

$$\text{area } ABMK = \text{unit area} \times l.$$

Again, in either figure, it is evident geometrically that the area $ABCD$ is derived from the area $ABMK$ by the same operations of partition and repetition as the line AD (b units) is derived from the line AK (unit length).

Thus we may write

$$\text{area } ABCD = \text{area } ABMK \times b.$$

Combining the two results

$$\textit{area } \boldsymbol{ABCD} = \textit{unit area} \times \boldsymbol{l} \times \boldsymbol{b},$$

where both multiplications are defined as in § 33.

36. Proportion.—The examples in §§ 34, 35, which are typical cases of multiplication, not only justify the definition of multiplication given in § 33, but also serve to show the intimate connection between multiplication and proportion.

Thus, in § 34 the operations of partition or repetition which are performed on the unit hour to obtain the given time indicate the *ratio* of that time to the unit hour; and since we perform the same operations on the distance 8 miles to find the distance travelled we are working on the principle that the ratio of the distance travelled to 8 miles is the same as the ratio of the time to 1 hour.

Now, an equality of ratios is a proportion, and thus the principle underlying the calculation is the proportion—

$$\text{required distance : 8 miles} = \text{given time : 1 hour.}$$

Similarly the principle underlying the formula in § 35 is a double application of the geometrical theorem that rectangles of the same altitude are proportional to their bases.

37. Multiplication and Division of Fractions.

Ex. *Evaluate* $$\frac{3}{7} \times \frac{2}{5}.$$

By definition (§ 33) this means perform the same operations on $\frac{3}{7}$ as are performed on unity to get $\frac{2}{5}$: that is, divide $\frac{3}{7}$ into 5 parts and take two of these parts.

Now $$\frac{3}{7} = \frac{3 \times 5}{7 \times 5} = 15 \text{ thirty-fifths.}$$

Divide into 5 parts: each is 3 thirty-fifths.

Take 2 of these: this gives 6 thirty-fifths.

Thus $$\frac{3}{7} \times \frac{2}{5} = \frac{6}{35}.$$

THEOREM V.—*To prove the rule for multiplication of fractions*, viz.

$$\frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs}.$$

By the definition of multiplication we must perform the operation represented by $\frac{r}{s}$ on the magnitude $\frac{p}{q}$.

Now $$\frac{p}{q} = \frac{ps}{qs}. \qquad [\textit{Theorem II.}$$

Thus the fraction on which we are to operate is represented by ps (qs)ths of the unit.

(i) Divide this group of ps fragments into s equal groups.
Each contains p (qs)ths of the unit.

(ii) Take r of these equal groups.
They contain between them pr (qs)ths of the unit.

Hence $$\frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs}.$$

THEOREM VI.—*To prove the rule for division of fractions*, viz.

$$\frac{p}{q} \div \frac{r}{s} = \frac{ps}{qr}.$$

To divide a by b is to find that number which when multiplied by b gives a.

Hence to prove this theorem it is only necessary to show that $\frac{ps}{qr}$ is the number which when multiplied by $\frac{r}{s}$ gives $\frac{p}{q}$.

But

$$\frac{ps}{qr} \times \frac{r}{s} = \frac{prs}{qrs} \qquad [\textit{Theorem V.}$$

$$= \frac{p}{q}. \qquad [\textit{Theorem II.}$$

This proves the theorem.

38. The Fundamental Laws.—In order to show that fractional numbers are subject to the ordinary rules of algebra we need only prove that they obey the Fundamental Laws (see §§ 12, 28).

I. *The Commutative Laws.*

(*a*) To prove that

$$\frac{p}{q}+\frac{r}{s}=\frac{r}{s}+\frac{p}{q}.$$

We have $\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}$ [§ 32

[ps and qr are integers] $=\frac{qr+ps}{qs}$ [§ 6

$=\frac{r}{s}+\frac{p}{q}.$ [§ 32

(*b*) To prove that

$$\frac{p}{q}\times\frac{r}{s}=\frac{r}{s}\times\frac{p}{q}.$$

We have $\frac{p}{q}\times\frac{r}{s}=\frac{p\times r}{q\times s}$ [§ 37

$=\frac{r\times p}{s\times q}$ [§ 9

$=\frac{r}{s}\times\frac{p}{q}.$ [§ 37

II. *The Associative Laws.*

(*a*) To prove that

$$\frac{p}{q}+\left(\frac{r}{s}+\frac{t}{u}\right)=\frac{p}{q}+\frac{r}{s}+\frac{t}{u}.$$

By § 32, $\frac{p}{q}=\frac{psu}{qsu},\quad \frac{r}{s}=\frac{qru}{qsu},\quad \frac{t}{u}=\frac{qst}{qsu}.$

Hence

$$\frac{p}{q}+\left(\frac{r}{s}+\frac{t}{u}\right)=\frac{psu}{qsu}+\left(\frac{qru}{qsu}+\frac{qst}{qsu}\right)$$

$=\frac{psu}{qsu}+\frac{(qru+qst)}{qsu}$ [§ 32

$=\frac{psu+(qru+qst)}{qsu}$ [§ 32

$=\frac{psu+qru+qst}{qsu}$ [§ 7

$=\frac{psu}{qsu}+\frac{qru}{qsu}+\frac{qst}{qsu}$ [§ 32

$=\frac{p}{q}+\frac{r}{s}+\frac{t}{u}.$

(*b*) To prove that

$$\frac{p}{q} \times \left(\frac{r}{s} \times \frac{t}{u}\right) = \frac{p}{q} \times \frac{r}{s} \times \frac{t}{u}.$$

We have

$$\frac{p}{q} \times \left(\frac{r}{s} \times \frac{t}{u}\right) = \frac{p}{q} \times \frac{rt}{su} \qquad [\S\ 37$$

$$= \frac{prt}{qsu} \qquad [\S\ 37$$

$$= \frac{pr}{qs} \times \frac{t}{u} \qquad [\S\ 37$$

$$= \frac{p}{q} \times \frac{r}{s} \times \frac{t}{u}. \qquad [\S\ 37$$

III. *The Distributive Law.*

To prove that

$$\frac{p}{q} \times \left(\frac{r}{s} + \frac{t}{u}\right) = \frac{p}{q} \times \frac{r}{s} + \frac{p}{q} \times \frac{t}{u}.$$

We have

$$\frac{p}{q} \times \left(\frac{r}{s} + \frac{t}{u}\right) = \frac{p}{q} \times \frac{ru + st}{su} \qquad [\S\ 32$$

$$= \frac{p\,(ru + st)}{qsu} \qquad [\S\ 37$$

$$= \frac{pru + pst}{qsu} \qquad [\S\ 11$$

$$= \frac{pru}{qsu} + \frac{pst}{qsu} \qquad [\S\ 32$$

$$= \frac{pr}{qs} + \frac{pt}{qu} \qquad [\S\ 32$$

$$= \frac{p}{q} \times \frac{r}{s} + \frac{p}{q} \times \frac{t}{u}. \qquad [\S\ 37$$

39. Fractional Numerators and Denominators.—The definition of fraction given in § 31 assumes that both numerator and denominator are integers, but we frequently have to calculate with fractions without knowing whether the letters in the numerators and denominators indicate integers or fractional numbers.

When p and q are not integers the symbol $\frac{p}{q}$ denotes $p \div q$.

This double meaning of the symbol $\frac{p}{q}$ would cause endless confusion but for the following important facts:—

(1) When p and q are integers the two interpretations give the same value, and

(2) These fractions with fractional numerators and denominators obey the same laws for reduction, addition, multiplication, etc., as ordinary fractions.

(1) To show that if p and q are integers the fraction $\frac{p}{q}$ is equal to the quotient $p \div q$, it is only necessary to show that $\frac{p}{q} \times q$ gives p (§ 19).

But $\frac{p}{q} \times q$ means q times p $q$ths,
which is the same as p times q $q$ths, [§ 9
which is the same as p times 1.

(2) This is dealt with in § 40.

40. Extension of Rules of Fractions.—That fractional numerators and denominators obey the same rules as integral can be proved from the fact that ordinary fractions obey the Fundamental Laws and therefore also the Derived Laws. In the following proofs then *the letters represent fractional numbers* :—

I. To prove that

$$\frac{p}{r} \pm \frac{q}{r} = \frac{p \pm q}{r}.$$

That is

$$p \div r \pm q \div r = (p \pm q) \div r.$$

By the definition of division it is sufficient to prove that the left-hand side of the equation when multiplied by r gives $p \pm q$.

Now

$$\begin{aligned} &(p \div r \pm q \div r) \times r \\ &= p \div r \times r \pm q \div r \times r && [\S\ 25 \\ &= \quad p \quad \pm \quad q. && [\S\ 20 \end{aligned}$$

II. To prove that $\frac{mp}{mq} = \frac{p}{q}.$

$$\begin{aligned}
\frac{mp}{mq} &= (m \times p) \div (m \times q) \\
&= m \times p \div m \div q && [\S\ 24 \\
&= p \times m \div m \div q && [\S\ 21 \\
&= p \div q && [\S\ 20 \\
&= \frac{p}{q}. && [\S\ 39
\end{aligned}$$

III. To prove that $\frac{p}{q} \pm \frac{r}{s} = \frac{ps \pm qr}{qs}.$

$$\begin{aligned}
\frac{p}{q} \pm \frac{r}{s} &= \frac{ps}{qs} \pm \frac{qr}{qs} && [\text{Case II.} \\
&= \frac{ps \pm qr}{qs}. && [\text{Case I.}
\end{aligned}$$

IV. To prove that $\frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs}.$

$$\begin{aligned}
\frac{p}{q} \times \frac{r}{s} &= (p \div q) \times (r \div s) \\
&= p \div q \times r \div s && [\S\ 24 \\
&= p \times r \div q \div s && [\S\ 21 \\
&= (p \times r) \div (q \times s) && [\S\ 24 \\
&= \frac{pr}{qs}.
\end{aligned}$$

V. To prove that $\frac{p}{q} \div \frac{r}{s} = \frac{ps}{qr}.$

$$\begin{aligned}
\frac{p}{q} \div \frac{r}{s} &= (p \div q) \div (r \div s) \\
&= p \div q \div r \times s && [\S\ 24 \\
&= p \times s \div q \div r && [\S\ 21 \\
&= (p \times s) \div (q \times r) && [\S\ 24 \\
&= \frac{ps}{qr}.
\end{aligned}$$

NOTE.—There is no need to prove that this new type of fractions obeys the fundamental laws, for the quotient $\frac{p}{q}$ is itself either an integer or a fraction, and both integers and fractions are already known to obey the fundamental laws.

CHAPTER IV.

POSITIVE AND NEGATIVE NUMBERS.

41. Positive and Negative Numbers.—Consider the equation

$$-7+4=-3.$$

We can give this a strictly arithmetical meaning, viz. that to subtract 7 from any number (not less than 7) and to add 4 to the remainder gives the same result as to subtract 3 from the original number.

If, however, we use the $+$ and $-$ signs to indicate opposite types of magnitude such as gains and losses, we can give other meanings to the equation, for example, that a loss of £7 together with a gain of £4 is equivalent to a net loss of £3. This is a typical case of the use of positive and negative numbers.

It is essential that the student should understand that we have not made any use of positive and negative *numbers* in the preceding chapters, though we have had plenty of positive and negative *terms*.

In positive and negative terms the $+$ and $-$ signs indicate addition and subtraction, as in all the preceding chapters, where the meaning of the different algebraic expressions can be expressed in ordinary arithmetical language.

In positive and negative *numbers* (sometimes called directed numbers) the $+$ and $-$ signs indicate opposite types of magnitude. Thus positive and negative numbers may be used to represent (1) credits and debts, (2) gains and losses, (3) movements north and movements south, etc.

This new meaning of the $+$ and $-$ signs may be regarded as the most characteristic distinction between algebraic and arithmetical methods of calculation.

Definition.—**Positive and negative numbers** *are numbers coupled with* $+$ *and* $-$ *signs used to denote opposite types of magnitude.*

It is generally advisable and sometimes necessary to distinguish positive and negative numbers by including them in brackets. Thus $(+6) - (-2)$ means subtract the negative number (-2) from the positive number $(+6)$.

Positive and negative numbers are also called **directed numbers.** In cases where they are applied to geometrical magnitudes the signs do indicate actual direction—east and west, or north and south, or up and down, etc.—but the name *directed numbers* is applied to all positive and negative numbers whatever types of magnitude they represent.

42. The Rules of Signs.—In § 18 we have proved the Rules of Signs for addition and subtraction. These can be represented symbolically as follows :—

$$+(+) = + \qquad\qquad -(+) = -$$
$$+(-) = - \qquad\qquad -(-) = +$$

In § 27 we have justified the Rules of Signs for multiplication and division.* These can be represented symbolically as follows :—

$$\begin{array}{l|l} (+) \times (+) = + & (+) \div (+) = + \\ (+) \times (-) = - & (+) \div (-) = - \\ (-) \times (+) = - & (-) \div (+) = - \\ (-) \times (-) = + & (-) \div (-) = + \end{array}$$

All these cases—in addition, subtraction, multiplication, and division—can be summarised in the one rule

like signs give $+$, unlike signs give $-$.

In evaluating an expression involving positive and negative numbers *we apply these same rules*, thus reducing the expression to one of arithmetical form which can be easily simplified. The validity of such a calculation will be discussed later : the first thing is to understand the method.

Ex. *Evaluate* $(+3) \times (-5) - (-4) \times (-7)$.

First perform the multiplications, using the rule of signs.

Thus $\quad$ Exp. $= (-15) - (+28)$.

Now remove the brackets, using the rule of signs.

$$\text{Exp.} = -15 - 28.$$

Evaluating this arithmetically we obtain the result, viz. -43.

---

* The Rules of Signs for division are easily deduced from those for multiplication.

Note that by an expression **in arithmetical form** we mean an expression in which *all brackets denoting positive or negative numbers have been removed* and which can therefore be evaluated by the ordinary rules for addition and subtraction of arithmetical terms.

43. Applications.—The most typical examples of the use of positive and negative numbers do not occur in elementary algebra but in allied branches of mathematics, notably **in the manipulation of formulae** in Trigonometry, Mechanics, and Coordinate Geometry.

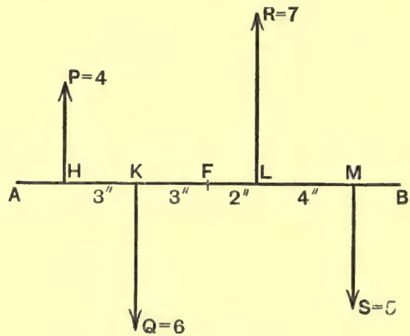

Fig. 6.

In order to ensure a clear understanding of the use of positive and negative numbers it is advisable in the first place to examine some particular instance. For this purpose we choose a calculation in the Theory of Moments which is quite easy to understand even if the student has read no Mechanics.

44. Theory of Moments.—Suppose that we have a horizontal bar AB pivoted at the point F, and acted upon by a series of vertical forces, P, Q, R, S, whose points of application are respectively H, K, L, M.

Each force tends to turn the bar round F: some tend to turn it clockwise and some counter-clockwise. It is required to determine which way the bar will turn, taking the dimensions given in the figure.

It is found that the "turning power" of any force in such a system is correctly measured by the product of the force and the distance of its point of application from F: thus the turning power of P is correctly measured by the product of $P \times HF$. This "turning power" is called the *moment* of the force, and the distance of the point of application from F is called the *arm* of the force: thus we have

$$moment = force \times arm.$$

Arithmetical Method.—To determine the resultant turning power (or moment) of the system *arithmetically* we should work as follows:—

Forces P and S both tend to turn the bar round F in the clockwise direction, and their moments are 4×6 and 5×6 respectively. Thus the total clockwise moment is $24 + 30$ or **54.**

Forces Q and R both tend to turn the bar counter-clockwise round F, and their moments are 6×3 and 7×2 respectively. Thus the total counter-clockwise moment is $18 + 14$ or **32.**

The result is a clockwise moment of $54 - 32$ or **22.**

Algebraic Method.—The algebraic method of working this calculation is based on the following *conventions* of signs:—

(i) Forces are reckoned positive when acting upwards, and negative when acting downwards.

(ii) Arms are reckoned positive when to the right of F, and negative when to the left of F.

(iii) Moments are reckoned positive when counter-clockwise, and negative when clockwise.

The calculation now becomes

$$\begin{aligned}
\text{Moment of } P &= (+4) \times (-6).\\
\text{Moment of } Q &= (-6) \times (-3).\\
\text{Moment of } R &= (+7) \times (+2).\\
\text{Moment of } S &= (-5) \times (+6).
\end{aligned}$$

Algebraic sum of moments of system

$$\begin{aligned}
&= (+4) \times (-6) + (-6) \times (-3) + (+7) \times (+2) + (-5) \times (+6)\\
&= \quad (-24) \quad + \quad (+18) \quad + \quad (+14) \quad + \quad (-30)\\
&= -24 + 18 + 14 - 30\\
&= -54 + 32 = -22.
\end{aligned}$$

This result, being negative, represents a moment of 22 in the clockwise direction.

45. Comparison of Methods.—If now we compare these two methods of calculation, the arithmetical and the algebraic, we see that in the former method we must examine the moment of each force separately to determine whether it is clockwise or counter-clockwise, whereas in the second case the clockwise and counter-clockwise moments are distinguished from each other by the mechanical application of the laws of signs for multiplication, and the resulting moment is then obtained by the mechanical application of the laws of signs for addition.

If all the forces (F_1, F_2, . . .) were upward and all the arms (a_1, a_2, . . .) rightward we should at once get the formula

$$\text{resulting moment} = F_1a_1 + F_2a_2 + F_3a_3 + \ldots$$

In other cases if we were working arithmetically we should have to write

$$\text{resulting moment} = \pm F_1a_1 \pm F_2a_2 \pm F_3a_3 \ldots$$

and we should have to determine which sign to give to each term by inspecting the figure.

Working algebraically (as in § 44) we still use the formula in the original form, i.e. *with all its signs* +, and substitute positive or negative numbers for the F's and a's according to the given convention. [Examine carefully the longest line in the algebraical calculation in § 44.]

It is this which constitutes the supreme value of directed (positive and negative) numbers. For any calculation which involves magnitudes that can be represented by positive and negative numbers we construct the formula by considering the simplest case, and we find that the formula so constructed *covers all the cases which occur*.

46. Explanation.—In order to demonstrate the validity of the algebraic method for the preceding calculation we must justify the use of the rules of signs in (i) the multiplication process and (ii) the addition process.

(i) *For Multiplication.* Note particularly that the + and − signs which we have assigned to the forces and arms are in no sense signs of addition and subtraction: in this calculation we are *not* adding or subtracting forces or arms, but only moments. The signs applied to the forces and arms are symbols of direction, and are in reality used only to determine the sign of the moment of the force.

Now the essential feature of the rule of signs in multiplication is that to reverse the sign of either factor reverses the sign of the product. We are justified in using this rule to determine the sign of the moment because if we reverse the direction of either force or arm we reverse the direction of the moment: *e.g.* an upward force with a leftward arm gives a clockwise moment, but if we reverse either the force or the arm we get a counter-clockwise moment.

If then the positive directions for force, arm and moment are so chosen that a positive force with a positive arm gives a positive moment, it will follow that the rule of signs for multiplication will give all the other cases correctly.

(ii) *For Addition.* The total moment is found from the separate moments by algebraical addition.

Now the essential feature of algebraic addition is that two terms cancel when they are equal in magnitude and opposite in sign: this corresponds exactly to the fact that two moments cancel if they are equal in magnitude and opposite in direction.

Thus the whole calculation is now justified.

47. Formal Algebra.—We are now in a position to explain the logical status of the laws of signs in algebra. These laws are applied (1) in the cases where the signs have their ordinary arithmetical meaning of addition and subtraction, as in Chapters I.-III., and (2) in the cases where the signs indicate positive and negative numbers, as in § 44. In the second case, however, the difficulty is that there are no general principles which cover all the applications of positive and negative numbers in the various mathematical formulae where they are used. *The different cases must be justified on their own merits* by some such method as we have used in § 46.

Under these circumstances the most convenient method of treatment from the logical point of view is to regard the laws of signs as *definitions.* That is to say, we base the methods of algebraic calculation on these laws of signs and we are then at liberty to apply these methods of calculation *in all cases where we can show independently that these laws are obeyed.** It is also

* Note that we have already shown that they are obeyed where the signs have their usual arithmetical meaning.

convenient to adopt the same plan with regard to the fundamental laws given in § 12, that is to regard them as definitions. This method of treatment is known as **formal algebra.**

Formal algebra may therefore be regarded as a science of calculation based on the fundamental laws and the laws of signs. The object of the science is to deduce from these laws the various processes and methods of calculation. These results can then be applied to any kind of numbers—integral, fractional, rational, irrational, positive, negative, etc.—*which can be shown independently to obey the laws on which formal algebra is based.*

48. The Fundamental Laws.—In formal algebra we assume by definition that positive and negative numbers obey the laws of signs; also that the numerical calculations involved—additions, subtractions, etc.—follow the ordinary rules. These two assumptions completely determine the nature and behaviour of positive and negative numbers.

For example

$$(-5)-(-3) = -5+3 \qquad \text{[Rule of Signs}$$
$$= -2.$$
$$(-5)\times(-3) = +(5\times 3) \qquad \text{[Rule of Signs}$$
$$= +15.$$

It remains therefore to demonstrate that positive and negative numbers so defined will obey the fundamental laws. If not, we should not be at liberty to apply the ordinary processes of calculation in the case of letters representing positive and negative numbers.

The method of proof is to use the rule of signs to reduce the expressions involving positive and negative numbers to expressions of arithmetical form involving only arithmetical numbers.

Ex. Verify the Distributive Law in the following case :—

$$(-5)\times\{(-3)+(+7)\} = (-5)\times(-3)+(-5)\times(+7).$$

Applying the rule of signs,

$$\text{L.H.S.} = (-5)\times\{-3+7\} = (-5)\times(+4) = -20;$$
$$\text{R.H.S.} = (+15)+(-35) = 15-35 = -20.$$

Proofs.—Suppose that a, b, c represent numbers which may be positive or negative. We can then write

$$a = (\pm x), \quad b = (\pm y), \quad c = (\pm z),$$

where x, y, z are arithmetical numbers.

Commutative Law for Addition.

$$a + b = (\pm x) + (\pm y) = \pm x \pm y.$$
$$b + a = (\pm y) + (\pm x) = \pm y \pm x.$$

Now the double signs indicate four possible combinations of signs. But in each case the signs of x and y will be the same in the two results.

Thus, since x and y are arithmetical, these two results are identical.

Hence $$a + b = b + a.$$

Commutative Law for Multiplication.

$$a \times b = (\pm x) \times (\pm y) = \pm x \times y,$$

where the last sign is determined by the usual rule of signs.

$$\begin{aligned} b \times a = (\pm y) \times (\pm x) &= \pm y \times x \\ &= \pm x \times y. \end{aligned}$$

By the rule of signs in each of the four possible cases the two results will have the same sign.

Hence $$a \times b = b \times a.$$

Associative Law for Addition.

$$\begin{aligned} a + (b + c) &= (\pm x) + \{(\pm y) + (\pm z)\} \\ &= \pm x + \{\pm y \ \pm z\} \\ &= \pm x \ \pm y \pm z. \end{aligned}$$

Also $$\begin{aligned} a + b + c &= (\pm x) + (\pm y) + (\pm z) \\ &= \pm x \pm y \pm z. \end{aligned}$$

By the rule of signs in each of the eight possible cases the two results will have the same signs.

Hence $$a + (b + c) = a + b + c.$$

Associative Law for Multiplication.

$$\begin{aligned} a \times (b \times c) &= (\pm x) \times \{(\pm y) \times (\pm z)\} \\ &= (\pm x) \times (\pm yz) \\ &= \pm xyz. \end{aligned}$$

Also $$a \times b \times c = (\pm x) \times (\pm y) \times (\pm z) = \pm xyz.$$

By the rule of signs in each of the eight possible cases the two results will have the same sign. Thus

$$a \times (b \times c) = a \times b \times c.$$

Distributive Law.

$$\begin{aligned} a \times (b + c) &= (\pm x) \times \{(\pm y) + (\pm z)\} \\ &= (\pm x) \times \{\pm y \pm z\} \\ &= \pm xy \pm xz. \\ ab + ac &= (\pm x)(\pm y) + (\pm x)(\pm z) \\ &= \pm xy \pm xz. \end{aligned}$$

By the rule of signs in each of the eight possible cases the two results will have the same signs. Thus

$$a \times (b + c) = ab + ac.$$

NOTE.—To understand the preceding proofs you must note that we assume the fundamental laws, and in some cases the derived laws, for arithmetical numbers—these have already been proved—and deduce the fundamental laws for directed numbers.

49. Negative Numbers in Elementary Algebra.—Negative numbers often occur as answers to algebraic problems, and as a rule an intelligible meaning can be assigned to them. In such cases it will be found that some (probably simple) formula has been used in solving the problem in which the use of positive and negative numbers can be justified; for example

gain = sale price − cost price,

total distance travelled northward

= sum of separate distances travelled northward.

The symbolism of negative number is often used in the transformation of formulae.

For example, since

$$\begin{aligned} (x+a)^3 &= x^3 + 3ax^2 + 3a^2x + a^3, \\ \therefore\quad (x-a)^3 &= \{x + (-a)\}^3 \\ &= x^3 + 3(-a)x^2 + 3(-a)^2x + (-a)^3 \\ &= x^3 - 3ax^2 + 3a^2x - a^3. \end{aligned}$$

This is justified as a convenient method of determining the alterations in sign produced by changing $+a$ to $-a$ in the evaluation of $(x+a)(x+a)(x+a)$.

In the same way the use of negative numbers in the formulae of Arithmetical and Geometrical Progressions and other series is merely a convenient method of determining the effect of performing certain numerical calculations with negative terms or multipliers instead of positive.

CHAPTER V.

APPLICATIONS OF POSITIVE AND NEGATIVE NUMBERS.

50. Verification of Laws and Signs in Formulae.—In any formula which involves the multiplication (or division) of positive and negative quantities, that is of quantities represented by positive and negative numbers, it is always easy to verify the **law of signs for multiplication** (or division) by the method used in § 46, starting from the fact that to reverse the sign of either factor (or of dividend or divisor) reverses the sign of the result. Note that as a rule the multiplier, multiplicand, and product (or divisor, dividend, and quotient) all represent different kinds of magnitude—for example, in § 46 they represent force, arm, and moment.

In the case of formulae involving the addition or subtraction of positive and negative quantities these quantities must, of course, represent the *same* kind of magnitude—we cannot add shillings to hours.

To verify the **laws of signs for addition and subtraction** in such cases is often difficult, but there are two general cases which are worth discussing, and which cover most, if not all, the special cases which actually occur: these are—

I. Where *the negative unit may be defined as that which cancels the positive unit.*

II. *The use of positive and negative coordinates.*

51. A Type of Negative Unit.—Many problems involve opposite types of magnitudes which tend to neutralise each other, such as gains and losses, or movements north and movements south. In order to deal with these we use positive and negative numbers.

In such cases a **negative unit** may be defined as a unit which cancels a positive unit; just as a loss of £1 cancels a gain of £1, or a movement one mile south cancels a movement one mile north.

Since in algebraical addition $+1$ cancels -1 we may represent the positive unit by $(+1)$ and the negative unit by (-1).

The expression $(+3)+(-5)$ accordingly means that we are to combine a group of 3 positive units with a group of 5 negative units. In this case the 3 positive units cancel with 3 of the negative units and 2 negative units remain, *i.e.*

$$(+3)+(-5)=(-2).$$

Addition.—It is easy to evaluate the *sum* of a set of positive and negative numbers. For example in the expression

$$(+3)+(-7)+(-6)+(+2) \quad \text{.......................}(a)$$

we have altogether 5 positive units and 13 negative units, which give 8 negative units as the total.

Now compare this process with the arithmetical calculation

$$+3-7-6+2=+5-13=-8 \quad \text{.....................}(b)$$

with which we are already familiar, and which means that to add 3, subtract 7, subtract 6, and add 2, gives the same ultimate result as to subtract 8. It is evident that *we use the same rule in both cases*, viz. sum the positive terms, sum the negative terms, take the numerical difference of the two results and prefix the sign of the larger result.

It is therefore convenient to evaluate the expression (a) by going through the calculation (b); that is to say for $+(-7)$ we write -7, for $+(+2)$ we write $+2$, and so on. This justifies the Laws for addition.

Ex. *How much is a man really worth if he owes* £15 *to A, B owes him* £8, *C owes him* £20, *and he owes* £19 to *D*.

Represent the amounts due to him as positive terms and his debts as negative terms.

$$\begin{aligned}&(-15)+(+8)+(+20)+(-19)\\&=-15+8+20-19=-34+28=-6.\end{aligned}$$

Thus he is ultimately £6 in debt, that is he will be able to pay all that he owes except £6.

Subtraction.—The rules for subtraction are proved by using the *fundamental* definition of subtraction, viz.—

> To subtract y from x means to find that number which when added to y gives x.

This is not the only meaning of subtraction, but it will be found that the other meanings are ultimately equivalent to this in all cases.

NOTE.—This definition applies in any problem which is solved by subtraction. For example—

P travels x miles east from London and Q travels y miles east from London. How far is P east of Q?

The answer is $x - y$.

Now the movements x and y may be either positive or negative, but in any case the problem is equivalent to the question—

What eastward movement must Q make to come up to P?

or, in other words—

What movement added to the movement y produces the movement x?

Ex. 1. *Evaluate* $(+3) - (-7)$.

What number added to (-7) gives $(+3)$?

Result: $(+10)$. For $(-7) + (+10) = (+3)$.

Ex. 2. *Evaluate* $(-5) - (-7)$.

What number added to (-7) gives (-5)?

Result: $(+2)$. For $(-7) + (+2) = (-5)$.

Taking the general case, if N be any number, positive or negative, it is required to evaluate

(i) $N - (+a)$ and (ii) $N - (-a)$.

(i) If $N + (-a)$ be added to $(+a)$ the result is N.

Thus, by the definition of subtraction,

$$N - (+a) = N + (-a)$$
$$= N - a. \qquad \text{[Proved}$$

(ii) If $N + (+a)$ be added to $(-a)$ the result is N.

Thus, by the definition of subtraction,

$$N - (-a) = N + (+a)$$
$$= N + a. \qquad \text{[Proved}$$

This justifies the Laws for subtraction.

The Double Negative.—It remains to consider the value of an expression such as $-(-a)$ when *both* signs indicate reversal.

$-(-a)$ then means the quantity which cancels a negative units, and this is evidently a positive units. So that here again we have

$$-(-a) = +a.$$

We have now established that *in all cases where the negative unit can be defined as that which cancels the positive unit* (see § 50 (I.)) *the quantities concerned obey the usual laws of signs for addition and subtraction.*

52. The Convention of Signs in Geometry.—One of the most important applications of positive and negative units is the algebraical method of dealing with the relative positions of different points on the same straight line.

In Fig. 7 let the division marks on PQ represent feet, and suppose that **displacements**—*i.e.* movements—to the right are considered positive and those to the left negative. (In this case the negative unit cancels the positive unit.)

Then
$$+3-5=-2$$
indicates that a displacement of 3 feet to the right followed by a displacement of 5 feet to the left produces a final displacement of 2 feet to the left; and so on.

Fig. 7.

Again, the fact that A is 3 feet to the right of B would be indicated by the equation
$$BA=+3,$$
for it is equivalent to the statement that the displacement BA is 3 feet to the right.

Also the fact that B is 3 feet to the left of A would be indicated by the equation
$$AB=-3,$$
for it is equivalent to the statement that the displacement AB is 3 feet to the left.

Thus $AB=-BA$, or $BA=-AB$; and so on.

It is evident that in whatever order A, B, C occur we should have
$$BC+CA+AB=0 \qquad \text{(i)}$$
for the three displacements BC, CA, AB would carry a point from B back to B. In Fig. 7
$$BC=-5,\ CA=+8,\ AB=-3,$$
hence the statement (i) corresponds to the equation
$$(-5)+(+8)+(-3)=0.$$

Similarly $BC+CA=BA$, and so on.

THEOREM.—*If A is m feet to the right of C and B is n feet to the right of C, prove that A is $(m-n)$ feet to the right of B; and show that this result holds for all values of m and n, positive or negative.*

[It is understood that m is a negative number when A is to the left of C, etc. Thus, in Fig. 8, $m = (-7)$, $n = (+3)$.]

In Fig. 7 m and n are both positive, and $m > n$. Evidently the distance of A to the right of B is $(m-n)$ feet.

Whether m and n are positive or negative numbers, it is given that displacement $CA = m$, and displacement $CB = n$: it is required to find the displacement BA.

Now in whatever order the points A, B, C occur, a point can move from C to A by the two displacements CB and BA. Thus

$$CB + BA = CA.$$

Hence by the definition of subtraction

$$\begin{aligned} BA &= CA - CB \\ &= m - n. \end{aligned}$$

Thus the distance of A to the right of B is $m - n$.

Fig. 8.

Ex. *A is p feet above C, and B is q feet above C. Construct a formula to determine how far A is above B. Then use the formula to determine the position of A relative to B in the following cases:—*

(i) A is 3 feet above C, B is 7 feet above C;

(ii) A is 5 feet above C, B is 3 feet below C;

(iii) A is 2 feet below C, B is 9 feet below C.

Assuming that p and q are both positive and that $p > q$ the problem reduces to simple arithmetic and we obtain the formula—

Distance of A above $B = p - q$ feet.

In constructing the formula we took each letter to represent the distance of one point *above* another. Hence we must reckon distance above a point as positive, and distance below a point as negative.

(i) If A is 3 feet *above* C, and B is 7 feet *above* C, then

$$p = +3, \quad q = +7.$$

Distance of A above $B = p - q = +3 - (+7) = +3 - 7 = -4$.

Result: A is 4 feet *below* B.

(ii) If A is 5 feet *above* C, and B is 3 feet *below* C, then

$$p = +5, \quad q = -3.$$

Distance of A above $B = p - q = +5 - (-3) = +5 + 3 = +8$.

Result: A is 8 feet *above* B.

(iii) If A is 2 feet *below* C, and B is 9 feet *below* C, then

$$p = -2, \quad q = -9.$$

Distance of A above $B = p - q = (-2) - (-9) = -2 + 9 = +7$.

Result: A is 7 feet *above* B.

Note particularly that the signs of the formula itself are *not* changed: the formula is $p - q$ all through. It is the signs of the numbers that are substituted for p and q that vary; also the meaning of the *result* depends upon its sign.

Exercises.

1. There are three mountains A, B, and C. A is x miles east of C, B is y miles east of C. How far is A east of B?

Use this formula to find A's position relative to B when

(i) A is 3 miles east of C, B is 7 miles east of C.

(ii) A is 5 miles east of C, B is 4 miles west of C.

(iii) A is 5 miles west of C, B is 4 miles west of C.

2. A trolley is under the influence of two forces, one h lb. wt. to the west, and the other k lb. wt. to the west. What single force would have the same effect as these two?

Use this formula to determine what single force would have the same effect as—

(i) 3 lb. wt. to the west and 7 lb. wt. to the east.

(ii) 3 lb. wt. to the east and 7 lb. wt. to the west.

(iii) 5 lb. wt. to the east and 5 lb. wt. to the west.

3. Two men start from the same point and run to the north. A takes p steps each r feet long, and B takes q steps each s feet long. How far is A north of B?

Use the formula to determine A's position relative to B when

(i) A has taken 100 steps each 5 feet to the north, and B has taken 80 steps each 6 feet to the south.

(ii) A has taken 50 steps each 7 feet to the south, and B has taken 60 steps each 6 feet to the north.

(iii) A has taken 80 steps each 6 feet to the south, and B has taken 70 steps each 5 feet to the south.

53. Positive and Negative Coordinates.—One of the most important applications of directed numbers is their use in Coordinate Geometry.

The position of any point P relative to two given axes, XOX', YOY', is determined by the coordinates OH, OK, where H and K are the projections of P on the two axes—oblique projections in the case of oblique axes.

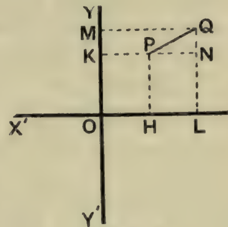

Fig. 9.

OH is reckoned positive or negative according as it lies right or left of O, and OK is reckoned positive or negative according as it lies above or below O.

Thus if in Fig. 9 the lengths OH, OK, OL, OM are 4, 5, 8, 7 units respectively, then the coordinates of P are (4, 5) and the coordinates of Q are (8, 7). In Fig. 10 if the lengths OH, OK, OL, OM are 4, 3, 5, 7 respectively, then the coordinates of P are $(-4, 3)$ and the coordinates of Q are $(5, -7)$.

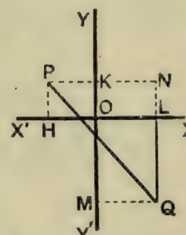

Fig. 10.

It is evident that the definition of a negative unit given in § 51 does not apply to negative coordinates. A length of 1 inch measured from O to the left does not cancel with a length of 1 inch measured from O to the right.

To justify the use of positive and negative numbers in this connection it is only necessary to prove the fundamental theorem on which Coordinate Geometry is based.

54. Relative Position.—The fundamental process in Coordinate Geometry is to determine the position of a point Q relative to a point P, when the coordinates of both points are given. See Figs. 9 and 10.

Now it is always possible to get from P to Q by two displacements PN, NQ, the first parallel to OX and the second parallel to OY. Using the convention of signs we see that in Fig. 9 both these displacements are positive, but that in Fig. 10 the displacement NQ is negative. In all cases *the position of Q relative to P is determined by the two displacements PN, NQ.*

55. The Fundamental Theorem of Coordinate Geometry.

THEOREM.—*If the coordinates of P are (x_1, y_1) and the coordinates of Q are (x_2, y_2), then whatever may be the signs of these coordinates the displacements PN, NQ are given in magnitude and direction by the equations*

$$PN = x_2 - x_1\ ;\ NQ = y_2 - y_1.$$

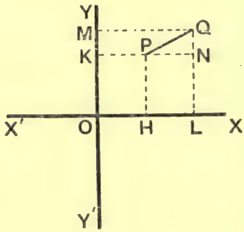

Fig. 9.

Proof.—Use Figs. 9 and 10.

Displacement PN = displacement HL.

With the usual conventions of signs (§ 52) whatever may be the relative positions of O, H, and L we have

$$OH + HL = OL\ ;$$

hence, by the definition of subtraction,

$$\begin{aligned} HL &= OL - OH \\ &= x_2 - x_1, \\ \therefore PN &= x_2 - x_1. \end{aligned}$$

Similarly

displacement NQ = displacement KM.

Also

$$\begin{aligned} OK + KM &= OM \\ \therefore KM &= OM - OK \\ &= y_2 - y_1 \\ \therefore NQ &= y_2 - y_1. \end{aligned}$$

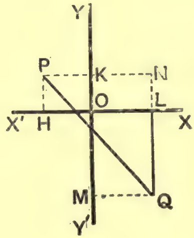

Fig. 10.

NOTE.—This theorem is evidently identical with that of § 52.

56. Trigonometry.—Many of the results in Coordinate Geometry are obtained by the use of Trigonometry, which introduces new sets of positive and negative units. A full discussion of these is a very difficult matter, but it is worth while to indicate the leading principles.

The Trigonometrical Ratios of any angle may be defined in terms of the projections of certain lines upon other lines.

Any one line in such a trigonometrical diagram has its own positive direction and the usual convention of signs applies to distances measured (or displacements) along the line.

The inclinations of the lines to one another are determined by means of a system of **angular coordinates.** If any line OX be taken as a reference line (Fig. 11), and if OP and OQ are any other lines through O, then the angles XOP and XOQ are the angular coordinates of OP and OQ. These angular coordinates are reckoned positive when they are traced counter-clockwise from OX.

If the angular coordinates of OP and OQ are θ_1 and θ_2, then the direction of OQ relative to OP is given by the angle $\theta_2 - \theta_1$: due regard being taken to the convention of signs. The proof of this is exactly the same as the proof of the theorem in § 55, except that we are dealing with angular rotations about a point instead of displacements along a line.

In Fig. 11, $\theta_1 = +32°$ and $\theta_2 = -48°$. Thus

$$\theta_2 - \theta_1 = (-48°) - (+32°) = -80°.$$

That is to say from OP to OQ is an angular rotation of 80° clockwise.

The same method determines the inclinations of any two lines in the plane if OP and OQ are drawn parallel to these lines and in the positive directions for these lines.

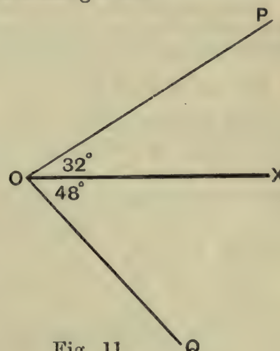

Fig. 11.

57. Dynamics.—In elementary dynamics the positive and negative units used in the formulae for motion in a straight line under uniform acceleration are of special interest.

I. $s = vt$.

This formula determines the distance s moved by a body in time t, under a uniform velocity v.

To fix our thoughts consider a special case.

Suppose we know that a motor-car running on a road pointing north and south passes through a village, P, at noon. Required to find s, its distance in miles north of P, at a time t hours after noon, given that it has a northward velocity of v miles per hour : s will be negative for distances south of P, t will be negative for times before noon, and v will be negative for southward velocities.

If we assume s, v, t all positive we obtain $s = vt$ as the required formula.

If s, v, t are not all positive it is easy to *verify* that the rule of signs for multiplication gives correct results in all cases.

Example. *Find the position of the car* 2 *hours before noon if the car is travelling* 25 *miles an hour southward.*

Here $\qquad v = -25,\ t = -2;$

hence $\qquad s = vt = (-25) \times (-2) = +50.$

Result ; 50 miles north of P — which is obviously correct.

In general it is evident that if in the data of the problem we change the sign of either v or t we change the sign of s, for the position of the car at a given time depends on whether that time is after noon or before noon, and

on whether the velocity of the car is northwards or southwards. Now the algebraic rule of signs for multiplication ensures that if the sign of either factor be reversed the sign of the product will be reversed. Hence the algebraic rule will give correct results in this problem.

Remembering that t can only have negative values if the motion considered has been going on before the instant from which t is reckoned, you will easily see that the above arguments apply in all cases.

II. $v = u + ft$.

This formula deals with a body moving in a straight line under the action of a constant force. The velocity at some given instant is u, and t seconds later it is v. This formula is not the result of abstract reasoning. It is *the mathematical statement of a law of nature,* being derived ultimately from Newton's second Law of Motion.

To fix our thoughts suppose that the line in which the motion is taking place runs east and west and that we are taking magnitudes (movements, velocities, etc.) directed eastwards as positive. Now the velocity at the end of the interval of t seconds is found to depend upon (i) the velocity at the beginning of the interval, called the *initial velocity,* and (ii) a velocity produced by the action of the force during the interval which we will call the *impressed velocity*. We may write

$$\text{final velocity} = \text{initial velocity} + \text{impressed velocity},$$

or
$$v \quad = \quad u \quad + \quad k$$

The impressed velocity, k, is directed east or west according as the force is directed east or west: also if the initial and impressed velocities are in opposite directions they tend to neutralise, *e.g.* if the initial velocity is 5 ft./sec. east and the impressed velocity is 8 ft./sec. west, the final velocity will be 3 ft./sec. west. We see then that in this case the negative unit of velocity may be defined as that which cancels the positive unit, and this justifies the use of the algebraic laws of addition in the formula.

The quantity f, called the acceleration, is the velocity impressed per second, its sign being positive or negative according as the force is directed eastward or westward. The term ft therefore correctly represents k, the impressed velocity in t seconds.

Lastly, if a negative value be assigned to t this changes the sign of the term ft and therefore gives the value of the velocity v at the corresponding time *before* the velocity is u.

III. $\boldsymbol{s = ut + \frac{1}{2} ft^2}$.

This formula determines the distance s travelled by a body in time t under a uniform acceleration, if the initial velocity is u and the acceleration is f.

The justification for the use of positive and negative numbers in this formula is that it is derived *algebraically* from formulae I. and II. which hold for positive and negative values.

Consider the usual proof of the formula. The time t is divided up into a very large number n of equal intervals i. The velocities at the beginning or end of these intervals are calculated by means of the formula $v = u + ft$, and then the space described during each interval is calculated by the formula $s = vt$. The final result $ut + \frac{1}{2} ft^2$ is the algebraic sum of these n distances. All these steps will be valid whether the quantities concerned are positive or negative.

It must be noted, however, that the distances described in some or all of these intervals may be negative, as the velocities for these intervals may be negative. Now if the total distance travelled in the positive direction be represented by $(+6)$ and the total distance travelled in the negative direction by (-10), s will be the algebraic sum $(+6) + (-10)$, *i.e.* (-4); whereas the total distance travelled reckoned arithmetically would be 16. It follows that s does not represent the total distance travelled in the arithmetic sense: it represents the *resulting displacement from the original position after it seconds.*

IV. $\boldsymbol{P = mf}$.

This formula determines the acceleration f produced in a mass m by a force P.

In this formula m is always positive, for negative mass has no meaning to us. Thus the algebraic rule of signs requires that P and f should always be of the same sign. This is evidently the case since the velocity impressed in one second by any force always has the same direction as the force.

The use of positive and negative quantities in the equations $\boldsymbol{S = \frac{1}{2}(u + v)\, t}$, $\boldsymbol{v^2 = u^2 + 2fs}$, $\boldsymbol{Pt = m(v - u)}$ and $\boldsymbol{Ps = \frac{1}{2}m(v^2 - u^2)}$ requires no further justification, as these are derived algebraically from the equations which have already been discussed.

CHAPTER VI.

IRRATIONAL NUMBERS.

58. Incommensurables.—Two quantities of the same kind are *commensurable* if it is possible to find a third quantity which is contained an exact number of times in each of them, *i.e.* if it is possible to find a *common measure* of the two quantities; if two quantities are commensurable the ratio of one to the other is either an integer or a vulgar fraction.

Any two quantities of the same kind, however, are not necessarily commensurable. For example—

The diagonal and the side of a square are incommensurable quantities.

Proof.—In sq. $ABCD$, let AE bisect $\angle DAC$, and let EF be $\perp$ AC.

Then it is easily proved that $AD = AF$, $DE = EF = FC$, and EC is the diagonal of a square whose side is DE.

If possible let some length L be a common measure of DC and AC, so that $DC = pL$, $AC = qL$, where p and q are integers.

Then

$$DE = FC = AC - AD = qL - pL = (q-p)L,$$

and

$$EC = DC - DE = pL - (q-p)L = (2p-q)L.$$

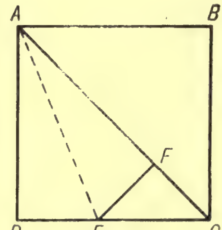

Fig. 12.

But $q - p$ and $2p - q$ are integers; hence L is also a common measure of DE and EC, *i.e.* of the side and diagonal of a square whose side is DE. Moreover, it is easy to show that $DE < \frac{1}{2}DC$.

By repeating the above argument it could then be shown that L is also a common measure of the side and diagonal of a square whose side is less than $\frac{1}{2}DE$, and therefore $< (\frac{1}{2})^2 DC$; and so on.

Thus ultimately L must be a common measure of the side and diagonal of a square whose side is $< (\frac{1}{2})^n DC$, where n may be any integer. But by increasing n sufficiently $(\frac{1}{2})^n DC$ may be made less than any assignable magnitude, and therefore less than L.

But L cannot be a measure of a quantity less than itself. Hence there is no common measure to the side and diagonal of a square.

59. The Concept of Number.—If two quantities are commensurable, we can express one as an exact fraction of the other: for example, if a third quantity is contained exactly p times in the first and exactly q times in the second, then the first quantity is p/q of the second. If, however, the two quantities are incommensurable we cannot express one as an exact fraction of the other: that is to say, if one of these two quantities is the unit of measurement, then the other cannot be expressed as an exact *number* (integral or fractional) of these units. Thus *the arithmetical concept of number is theoretically incomplete.*

Let us look more carefully into this question. Suppose the line AB (Fig. 13) contains 3 units of length, and suppose that a point P travels from A to B. Then we conceive the distance AP as increasing *continuously* from zero to 3 units: that is to say, the distance AP passes through all possible magnitudes between zero and AB; and yet if we use

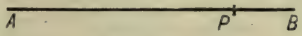

Fig. 13.

only the arithmetical concept of number we have no means of expressing the distance AP accurately in terms of the unit of length whenever P is in such a position that AP is incommensurable to the unit of length. Accordingly we require somehow to extend our conception of number in such a way that number (like magnitude) may be supposed to increase *continuously* from any one value to any other.

Thus we must *assume** that numbers exist which cannot be accurately expressed by integers or fractions, and that any two incommensurable quantities have a ratio expressed by means of such a number.

60. Irrational Numbers.—All numbers which can be expressed by integers or fractions (proper or improper) are called **rational** numbers, and all numbers which cannot be so expressed are called **irrational.**

The object of this chapter is to give a logical account of irrational numbers. It will be necessary—

(i) to explain how an irrational number is specified;

(ii) to explain how irrational numbers can be subjected to the

* The existence of such numbers can be proved from simpler premises —see Russell's *Principles of Mathematics*,—but the discussion is far too difficult for an elementary book.

ordinary operations of arithmetic, such as addition, subtraction, multiplication, etc.; and

(iii) to prove that these operations can be applied to irrational numbers under the same rules as to rational.

Function of Number.—The function of number is primarily to enable us to compare magnitudes—to say whether two different magnitudes are equal, or which of them is the greater. In other words, number enables us to arrange different quantities in "order of magnitude," or in **sequence,** as it is sometimes called.

Now it would be quite easy to arrange a number of different straight lines (for example) in sequence by purely geometrical tests, without the use of numbers; and it is obvious that in so doing the question as to whether these lines were or were not commensurable to one another or to some unit would not occur. Accordingly we must assume that all numbers, whether rational or irrational, are subject to the concept of sequence, or order of magnitude, and that all ideas derived from this concept (such as the idea of one number being intermediate in value to two others, etc.) may be applied equally to rational or irrational numbers.

We shall find that this concept of sequence as applicable to both rational and irrational numbers is sufficient basis for a logical theory of irrational numbers.

61. Specification of an Irrational Number.—In Fig. 13 suppose that we wish to express AP in centimetres, and that AP is incommensurable to the centimetre. Imagine that AB is divided up into hundredths of centimetres starting from A: then if P lies between the 236th and 237th points of division the length of AP in centimetres lies between 2·36 and 2·37. Imagine, again, that AB is divided up into thousandths of centimetres starting from A: then if P lies between the 2364th and 2365th points of division the length of AP in centimetres lies between 2·364 and 2·365.

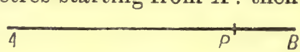

Fig. 13.

Similarly, by supposing that AB is divided up into (i) ten-thousandths of centimetres, (ii) hundred-thousandths of centimetres, (iii) millionths of centimetres, and so on, we should find (say) that the length of AP in centimetres lies between (i) 2·3640 and 2·3641, (ii) 2·36405 and 2·36406, (iii) 2·364057 and 2·364058, and so on. Thus we see that the length of AP in centimetres can be specified as a number which *lies between two rational numbers* which can be determined by a certain definite process, and the *difference* between which can ultimately be made *as small as we please.*

Again, suppose we wish to indicate exactly what is meant by the irrational number $\sqrt{7}$. It can be shown that $2{\cdot}645^2 < 7$, and $2{\cdot}646^2 > 7$; that $2{\cdot}6457^2 < 7$, and $2{\cdot}6458^2 > 7$; and so on. Thus the irrational number $\sqrt{7}$ lies between 2·645 and 2·646 and between 2·6457 and 2·6458, and so on.

Hence $\sqrt{7}$ may be specified as a number which lies between any two numbers whose squares are respectively greater and less than 7.

Thus we see that **an irrational number can be specified as a number which always lies between two rational numbers, which can be determined by some definite process such that the difference between them can ultimately be made as small as we please.**

It may be well to note at this point that the present discussion is concerned entirely with positive quantities. The concept of number is arithmetical and is quite independent of the algebraical question of sign: when the arithmetical concept of number is completed by the theory of irrational numbers, then the use of the + and − signs can be adopted without further trouble for all numbers.

62. Notation.—In the remainder of this discussion it will be convenient to use small letters to denote rational numbers only, and to use capital letters to denote numbers (usually, but not necessarily, irrational) which are specified as always lying between two rational numbers.

Also when a number A is specified as lying between two rational numbers a and a', it will always be assumed that the second number a' is greater than the first number a. Notice particularly that a and a' do not represent *fixed* rational numbers but *variable* rational numbers, and that the difference $a' - a$ ultimately vanishes, that is ultimately becomes indefinitely small.

63. Arithmetical Operations with Irrational Numbers.—Irrational numbers will obviously be of no value to us unless we can apply to them the ordinary arithmetical operations of addition, multiplication, etc. But the usual *definitions* of these operations apply only to rational numbers, and we must accordingly make special definitions applicable to irrational numbers.

DEFINITIONS.—*Suppose we have two irrational* numbers A and B, which are specified as always lying between a and a', and between b and b' respectively. Then—*

I. **The** *sum* $A + B$ **is the number which always lies between the rational numbers** $(a + b)$ **and** $(a' + b')$.

II. **The** *difference* $A - B$ **is the number which always lies between the rational numbers** $(a - b')$ **and** $(a' - b)$.†

III. **The** *product* $A \times B$ **is the number which always lies between the rational numbers** ab **and** $a'b'$.

IV. **The** *quotient* $A \div B$ **is the number which always lies between the rational numbers** $a \div b'$ and $a' \div b$.‡

64. Vanishing Differences.—Before these definitions can be admitted it is obviously necessary to show that the difference between the two rational numbers used in any definition ultimately vanishes—that is, can be made as small as we please.

Given that the difference between a and a' can be made as small as we please, and also the difference between b and b',

Required to prove that the differences between—

I. $(a + b)$ and $(a' + b')$, III. $(a \times b)$ and $(a' \times b')$,
II. $(a - b')$ and $(a' - b)$, IV. $(a \div b')$ and $(a' \div b)$,

can each be made as small as we please.

Proof.—Let $a' = a + h$, $b' = b + k$, then h and k can be made as small as we please.

I. $(a' + b') - (a + b) = h + k$, which ultimately vanishes.

II. $(a' - b) - (a - b') = h + k$, which ultimately vanishes.

III. $(a' \times b') - (a \times b) = (a + h)(b + k) - (a + b) = bh + ak + hk$, which can be made as small as we please by diminishing h and k, since a and b do not tend to become indefinitely large.

IV. $\frac{a'}{b} - \frac{a}{b'} = \frac{a'b' - ab}{bb'} = \frac{(a + h)(b + k) - ab}{b(b + k)} = \frac{ak + bh + hk}{b(b + k)}$, which can be made as small as we please by diminishing h and k, since a and b do not tend to become indefinitely large or indefinitely small.

65. Reverse Processes.—In order that the definitions of § 63 may be admitted as equivalent for irrational numbers to the ordinary definitions for rational numbers, it is obviously necessary to show that the process of

---

* The case where one number is irrational and the other rational is simpler than the case where both are irrational, and the theory for the former case is obvious enough when the latter is mastered.

† That is, between the least and the greatest of the four numbers $a - b$, $a' - b$, $a - b'$, $a' - b'$.

‡ That is, between the least and the greatest of the four numbers $a \div b$, $a \div b'$, $a' \div b$, $a' \div b'$.

subtraction for irrational numbers is the reverse of the process of addition, and that division is the reverse of multiplication.

I. **For irrational numbers subtraction is the reverse of addition.**

Given that A always lies between a and a',
B ,, ,, ,, b and b',
and $A - B$,, ,, ,, $(a - b')$ and $(a' - b)$, [Def. II.

Required to prove that $B + (A - B) = A$.

Proof.—Let $a' = a + h$, $b' = b + k$.

Then B always lies between b and $(b + k)$,
while $(A - B)$,, ,, ,, $(a - b - k)$ and $(a + h - b)$.

Thus, by Def. I.—
$B + (A - B)$,, ,, ,, $(a - k)$ and $(a + h + k)$ ....(i)

Now A always lies between a and $a + h$;
also both a and $(a + h)$ lie ,, $(a - k)$ and $(a + h + k)$;
Hence A always lies ,, $(a - k)$ and $(a + h + k)$ ......(ii)

From (i) and (ii) it follows that the two numbers $B + (A - B)$ and A always lie between the same two rational numbers whose difference can be made as small as we please. Hence $B + (A - B) = A$.

II. **To prove that for irrational numbers division is the reverse of multiplication.**

Given that A always lies between a and a',
B ,, ,, ,, b and b',
$(A \div B)$,, ,, ,, $\frac{a}{b'}$ and $\frac{a'}{b}$, [Def. IV.

Required to prove that $B \times (A \div B) = A$.

Proof.—Let $a' = a + h$, $b' = b + k$.

Then B always lies between b and $b + k$,
and $(A \div B)$,, ,, ,, $\frac{a}{b+k}$ and $\frac{a+h}{b}$,

Therefore by Def. III.—
$B \times (A \div B)$,, ,, ,, $\frac{ab}{b+k}$ and $\frac{(a+h)(b+k)}{b}$ ......(i)

Now A always lies between a and $(a + h)$;

also obviously $a > \frac{ab}{b+k}$ $\quad$ (for $\frac{b}{b+k} < 1$),

and $(a + h) < \frac{(a+h)(b+k)}{b}$ $\quad$ (for $\frac{b+k}{b} > 1$);

Hence A always lies between $\frac{ab}{b+k}$ and $\frac{(a+h)(b+k)}{b}$ ....(ii)

From (i) and (ii) it follows that the two numbers $B \times (A \div B)$ and A always lie between the same two rational numbers whose difference can be made as small as we please. Hence $B \times (A \div B) = A$.

66. The Fundamental Laws.—In order that the preceding theory of irrational numbers should be complete, it remains to show that these irrational numbers obey the three fundamental Laws, viz. the Commutative Laws, Associative Laws, and the Distributive Law.

I. The Commutative Laws.

To prove that

$$B + A = A + B, \text{ and } B \times A = A \times B.$$

Proof.—$B + A$ always lies between $b + a$ and $b' + a'$,
while $A + B$,, ,, ,, $a + b$ and $a' + b'$.
Also $B \times A$,, ,, ,, ba and $b'a'$,
while $A \times B$,, ,, ,, ab and $a'b'$.

But

$$b + a = a + b, \; b' + a' = a' + b', \; ba = ab, \text{ and } b'a' = a'b',$$

for the Commutative Laws hold for all rational numbers.

Thus $A + B$ and $B + A$ always lie between the same two rational numbers whose difference ultimately vanishes. Hence

$$A + B = B + A.$$

Similarly $$A \times B = B \times A.$$

II. Associative Laws.

To prove that

$$A + (B + C) = A + B + C,$$

and

$$A \times (B \times C) = A \times B \times C.$$

Proof.—By Def. I. $(B + C)$ always lies between $(b + c)$ and $(b' + c')$.

Hence by Def. I. $A + (B + C)$,, ,, ,, $a + (b + c)$ and $a' + (b' + c')$, *i.e.* between

$$a + b + c \text{ and } a' + b' + c' \quad \text{(i)}$$

for the Associative Laws hold for all rational numbers.

Also by Def. I. $A + B + C$ always lies between

$$a + b + c \text{ and } a' + b' + c' \quad \text{(ii)}$$

From (i) and (ii) it follows that $A + (B + C)$ and $A + B + C$ always lie between the same two rational numbers whose difference ultimately vanishes.

Hence

$$A + (B + C) = A + B + C,$$

A similar argument shows that

$$A \times (B \times C) \text{ and } A \times B \times C \text{ both lie between}$$
$$a \times b \times c \text{ and } a' \times b' \times c'.$$

Hence

$$A \times (B \times C) = A \times B \times C.$$

III. THE DISTRIBUTIVE LAW.

To prove that

$$A \times (B + C) = A \times B + A \times C.$$

Proof.—By Def. I. $(B + C)$ always lies between $b + c$ and $b' + c'$;

Thus, by Def. III.—

$$A \times (B + C) \text{ always lies between } a(b + c) \text{ and } a'(b' + c') \quad \text{..........(i)}$$

By Def. III. $A \times B$ always lies between ab and $a'b'$,
and $A \times C$ „ „ „ ac and $a'c'$;
hence, by Def I., $A \times B + A \times C$ always lies between

$$ab + ac \text{ and } a'b' + a'c',$$

i.e. between

$$a(b + c) \text{ and } a'(b' + c') \quad \text{..............................(ii)}$$

for the Distributive Law holds for all rational numbers.

From (i) and (ii) it follows that $A \times (B + C)$ and $A \times B + A \times C$ always lie between the same two rational numbers whose difference ultimately vanishes.

Hence

$$A \times (B + C) = A \times B + A \times C.$$

67. Product of Surds. *Prove that* $\sqrt{2} \times \sqrt{3} = \sqrt{6}$.

$\sqrt{2}$ is specified as a number which lies between a and a', where $a^2 < 2$, $a'^2 > 2$, and $a' - a$ can be made as small as we please.

$\sqrt{3}$ is specified as a number which lies between b and b', where $b^2 < 3$, $b'^2 > 3$, and $b' - b$ can be made as small as we please.

By Def. III. $\sqrt{2} \times \sqrt{3}$ always lies between ab and $a'b'$.

But $a^2b^2 < 6$, $a'^2b'^2 > 6$.

Hence $\sqrt{2} \times \sqrt{3}$ always lies between two rational numbers whose difference can be made as small as we please, such that the square of the smaller number is always less than 6 and the square of the larger number greater than 6. But this is the specification of $\sqrt{6}$.

The preceding argument will evidently apply in all similar cases. Hence we have $\sqrt[n]{x} \times \sqrt[n]{y} = \sqrt[n]{xy}$.

68. THEOREM.—*The area of a rectangle is given by the product of the measures of its sides, when the sides are incommensurable with the unit of length.*

In rect. $ABCD$, let AB and AD contain X and Y units of length respectively, where X and Y are irrational.

Let B lie between two points b and b' in AB, and let D lie between two points d and d' in AD, such that Ab, Ab', Ad, Ad', contain x, x', y, y' units of length, where $x < x'$, $y < y'$, and these numbers are all rational.

Complete the rects. $Abcd$, $Ab'c'd'$. Then it follows from the figure that

$$\text{rect. } Abcd < \text{rect. } ABCD < \text{rect. } Ab'c'd'.$$

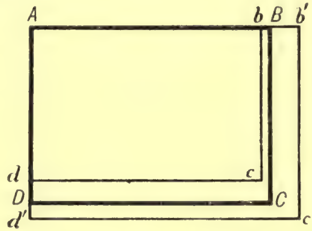

Fig. 14.

Now the difference between x and x' and the difference between y and y' can be made as small as we please. Also X lies between x and x', and Y lies between y and y'.

Hence XY is specified as the number which always lies between

$$xy \text{ and } x'y' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Again, measure of area $Abcd$ is xy, and measure of area $Ab'c'd'$ is $x'y'$. Also area $ABCD$ always lies between area $Abcd$ and area $Ab'c'd'$. Hence

$$\text{measure of area } ABCD \text{ always lies between } xy \text{ and } x'y'. \quad \ldots\ldots\text{(ii)}$$

From (i) and (ii) it follows that

$$\text{measure of area } ABCD = XY.$$

69. Irrational Exponent.—We have now completed the theory of irrational numbers, except for the case when the irrational is used as an index. The *method* of treatment is essentially the same as before, though the details of the discussion are more intricate.

Exercises.

1. How would you define $A \times b$, where A is irrational and b rational?

2. How would you define $a \div B$, where a is rational and B irrational? Prove from your definition that $(a \div B) \times B$ is equal to a.

3. How would you define A^b where A is irrational and b integral? Prove that $A^b \times A^c = A^{b+c}$.

4. Prove that $\sqrt[3]{p} \div \sqrt[3]{q} = \sqrt{\dfrac{p}{q}}$ where p and q are rational, and $\sqrt[3]{p}$ and $\sqrt[3]{q}$ are irrational.

CHAPTER VII.

THE THEORY OF INDICES.

70. Positive Integral Indices.

DEFINITION I.—*If* n *is a positive integer,* a^n *represents* $a \times a \times a \times \ldots$ **to n factors.**

Operations involving indices are governed by the three Index Laws, which are proved as follows:—

LAW I. $\begin{cases} a^m \times a^n = a^{m+n} \\ a^m \div a^n = a^{m-n}, \text{ if } m > n. \end{cases}$

Proofs.—

(i) $a^m \times a^n = (a\,.\,a\,.\,a \ldots \text{ to } m \text{ factors}) \times (a\,.\,a \ldots \text{ to } n \text{ factors})$ [*Def.*

$= a\,.\,a\,.\,a\,.\,a\,.\,a \ldots \text{ to } (m+n) \text{ factors}$ [*Assoc. Law.*

$= a^{m+n}.$ [*Def.*

(ii) $a^m \div a^n = \dfrac{a\,.\,a\,.\,a \ldots \text{ to } m \text{ factors}}{a\,.\,a \ldots \text{ to } n \text{ factors}}$ [*Def.*

$= a\,.\,a \ldots \text{ to } (m-n) \text{ factors}$

$= a^{m-n}.$ [*Def.*

COROLLARY.—$a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\cdots}$.

LAW II. $(a^m)^n = a^{mn}$.

Proof.—

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \ldots \text{ to } n \text{ factors} && [\textit{Def.}\\ &= a^{m+m+m+\cdots \text{ to } n \text{ terms}} && [\text{By Law I.}\\ &= a^{m\times n}\\ &= a^{mn}.\end{aligned}$$

COROLLARY.—$\left\{(a^m)^n\right\}^p = a^{mnp}$, etc.

LAW III. $\left(\dfrac{abc\ldots}{def\ldots}\right)^n = \dfrac{a^n b^n c^n \ldots}{d^n e^n f^n \ldots}.$

Proof.— $\left(\dfrac{abc\ldots}{def\ldots}\right)^n = \dfrac{abc\ldots}{def\ldots} \times \dfrac{abc\ldots}{def\ldots} \times \ldots \text{ to } n \text{ factors}$

$= \dfrac{a^n b^n c^n \ldots}{d^n e^n f^n \ldots}.$

COROLLARY.—$(abc\ldots)^n = a^n b^n c^n \ldots.$

71. Fractional, Zero, and Negative Indices.—Throughout the whole of § 70 it has been assumed that the indices are positive integers. This indeed is inevitable, because the definition given there has *no meaning* if the index n is not a positive integer: for example, it does not enable us to assign a meaning to such expressions as $a^{\frac{3}{5}}$, a^0, a^{-5}.

The question now arises whether this restriction can be removed, that is—whether the use of fractional, zero, or negative indices is possible and advisable. If we are to use them it is evident that *new definitions* will be required.

We shall find that the removal of this restriction is not only possible, but is one of the most important developments in elementary algebra. The meanings to be assigned to fractional, zero, and negative indices are given by the following additional definitions, which are fully discussed in subsequent articles.

ADDITIONAL DEFINITIONS.—

II. $a^{\frac{p}{q}}$ **means** $\sqrt[q]{a^p}$, where p and q are any positive integers.

III. a^0 **means 1.**

IV. a^{-n} **means** $\dfrac{1}{a^n}$, where n is any positive number.

72. Necessary Conditions.—In the ordinary way one cannot be required to prove, or to give any logical justification for, a definition. For example, there is no possible proof of the definition of a circle: it is merely a statement of the precise meaning which mathematicians attach to the word. Again, if we choose to define $| x | y |$ as a symbol representing $\sqrt[3]{(x^3+x^2y+xy^2+y^3)}$ we are logically quite at liberty to do so. It merely remains to be seen whether the symbol is of any practical value.

It might appear then that we have a logical right to adopt the new definitions given in § 71 without further discussion. But in this case the position is altogether different, for the reason that we are not defining entirely *new* symbols: we have already one definition of a power, and three Index Laws based on that definition, and their existence introduces practical considerations which are of fundamental importance.

For example, the index in $a^{\frac{20}{4}}$ may be regarded either as a fraction or as an integer, so that we should evidently be obliged to reject any definition of $a^{\frac{p}{q}}$ which did not make $a^{\frac{20}{4}}$ equal to a^5.

Again, if we have to evaluate $a^x \times a^y$ we know that if x and y are positive integers the result will be a^{x+y}. Now it would introduce intolerable confusion if the result were something different when x and y were fractional or negative: for we may have no means of knowing whether x and y are positive integers or not.

We may infer then that we are at liberty to adopt any definitions of fractional, zero, and negative indices provided that

(i) *these definitions are consistent with the definition of positive integral indices*, and

(ii) *the indices so defined will obey the same three Index Laws as positive integral indices.*

It would seem possible that these conditions might still leave us some choice in framing the new definitions. It will be found, however, that if we make only the *single* assumption (in a few special cases) that all indices are to obey the First Index Law we shall be forced to adopt the above definitions as the only possible ones. Having obtained the definitions in this way, it is then necessary to verify that they fully satisfy the conditions just laid down.

The complete investigation is therefore as follows:—

(*a*) Assuming that the First Index Law is to hold for all indices, deduce the definitions (§ 73).

(*b*) Show that these definitions are consistent with the original definition (§ 75).

(*c*) Show that the new indices, so defined, obey the same Index Laws as positive integral indices (§§ 76-79).

The theory of indices will then be established on the basis of Definitions I.-IV. (§§ 70, 71) and Laws I.-III. (§ 70). These Definitions and Laws will be referred to by these numbers.

73. To find meanings for (**i**) $a^{\frac{p}{q}}$, (**ii**) a^0, (**iii**) a^{-n}, assuming that the First Index Law is to hold for all indices.

(i) If p and q are positive integers,

$$\left(a^{\frac{p}{q}}\right)^q = a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \ldots \text{ to } q \text{ factors} \quad [\textit{Def. I.}$$

$$= a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\ldots \text{ to } q \text{ terms}} \quad [\textit{Law I.}$$

$$= a^{\frac{p}{q}\times q} = a^p.$$

Thus the qth power of $a^{\frac{p}{q}}$ is a^p.

Hence $a^{\frac{p}{q}}$ is the qth root of a^p,

i.e. $$\boldsymbol{a^{\frac{p}{q}} = \sqrt[q]{a^p}}.$$

(ii) $$a^n \times a^0 = a^{n+0} \qquad [\textit{Law I.}$$
$$= a^n.$$

Hence
$$a^0 = a^n \div a^n,$$

i.e. $$\boldsymbol{a^0 = 1}.$$

(iii) If n is any positive number, integral or fractional,
$$a^n \times a^{-n} = a^{n-n} \qquad [\textit{Law I.}$$
$$= a^0$$
$$= 1.$$

Hence $$\boldsymbol{a^{-n} = \frac{1}{a^n}}.$$

We have now shown that if all indices are to obey the First Index Law, Definitions II., III., IV. (§ 71) are the only possible definitions for fractional, zero, and negative indices.

In Definition II. note that the numerator of the index represents a power and the denominator a root.

NOTE.—When we are extracting an even root of any number there is an ambiguity of sign; for example, the fourth root of 16 is ± 2. It would, however, be intolerable to use symbols whose meaning was ambiguous: accordingly all such expressions as $\sqrt[4]{16}$, $81^{\frac{3}{4}}$, $a^{\frac{5}{8}}$, etc., are to be interpreted with the positive sign. If the negative root is intended the negative sign must be prefixed.

74. Order of Operations.—If it is required to evaluate $a^{\frac{p}{q}}$ where a, p, and q are given numbers we must use the definition $a^{\frac{p}{q}} = \sqrt[q]{(a^p)}$. Thus we should find the pth power of a, and then the qth root of the result.

It will usually be more convenient to *reverse the order of the operations*, i.e. to find the qth root of a and then the pth power of the result. Thus we should write

$$a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p.$$

This reversal in the order of operations is justified by means of § 67. For

$$\begin{aligned}(\sqrt[q]{a})^p &= \sqrt[q]{a} \times \sqrt[q]{a} \times \sqrt[q]{a} \times \ldots \text{ to } p \text{ factors} \\ &= \sqrt[q]{(a \times a \times a \times \ldots \text{ to } p \text{ factors})} \qquad [§ 67 \\ &= \sqrt[q]{a^p}.\end{aligned}$$

75. Proofs of Index Laws.—It now remains to prove that the new definitions of indices are consistent with the original definitions and also that the three* Index Laws can be established for all indices, integral, zero, and fractional, positive and negative.

It is easy to show that there is no inconsistency between the new definitions and the original one, for there is only one case to be considered.

The index 0 cannot represent a positive integer, nor can the index $-n$, if n is positive. But the index $\frac{p}{q}$ will reduce to a positive integer, say r, if $p = qr$.

Using the definition of a fractional index we have in this case

$$a^{\frac{p}{q}} = a^{\frac{qr}{q}} = \sqrt[q]{a^{qr}} = a^r.$$

Hence the fractional index definition gives the correct value of $a^{\frac{p}{q}}$ when $\frac{p}{q}$ reduces to an integer.

76. To prove that $a^m \times a^n = a^{m+n}$ for all values of m and n.

CASE I.—*Where m and n are positive integers.*

This has already been proved in § 70.

CASE II.—*Where m and n are positive fractions.*

Let m and n be denoted by $\frac{p}{q}$ and $\frac{r}{s}$ respectively, where p, q, r and s are positive integers.

---

* We have shown that the new definitions can be obtained by assuming the First Index Law in certain special cases. It is still necessary to show that if these definitions are adopted the First Index Law holds in *all possible cases.*

Then $$a^m \times a^n = a^{\frac{p}{q}} \times a^{\frac{r}{s}} = k, \text{ suppose.}$$
Also qs is a positive integer.

$$\therefore\ k^{qs} = \left(a^{\frac{p}{q}} \times a^{\frac{r}{s}}\right)^{qs} = \left(a^{\frac{p}{q}}\right)^{qs} \times \left(a^{\frac{r}{s}}\right)^{qs} \qquad \text{[}\textit{Law III. for Pos. Int.}$$
$$= \left\{\left(a^{\frac{p}{q}}\right)^q\right\}^s \times \left\{\left(a^{\frac{r}{s}}\right)^s\right\}^q \qquad \text{[}\textit{Law II. for Pos. Int.}$$
$$= \{(\sqrt[q]{a^p})^q\}^s \times \{(\sqrt[s]{a^r})^s\}^q \qquad \text{[}\textit{Def. II.}$$
$$= (a^p)^s \times (a^r)^q \qquad \text{[}\textit{Def. of Root}$$
$$= a^{ps} \times a^{qr} = a^{ps+qr} \qquad \text{[}\textit{Laws I., II. for Pos. Int.}$$

Hence $$k = \sqrt[qs]{a^{ps+qr}}.$$

Thus by definition of fractional indices

$$k = a^{\frac{ps+qr}{qs}} = a^{\frac{ps}{qs}+\frac{qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}} = a^{m+n},$$

which proves the law for positive fractional values of m and n.

CASE III.—*Where one of the indices, say n, is negative.*

Let $n = -p$, where p is positive.

(i) If $m - p$ is positive,

$$a^{m-p} \times a^{-p} = a^m, \qquad \text{[Case I. or II.}$$
$$\therefore\ a^{m-p} = a^m \times \frac{1}{a^p} = a^m \times a^{-p}, \qquad \text{[}\textit{Def. IV.}$$

i.e. $$a^{m+n} = a^m \times a^n.$$

(ii) If $p - m$ is positive,

$$a^{p-m} \times a^m = a^p \qquad \text{[Case I. or II.}$$
$$\therefore\ \frac{1}{a^{p-m}} = a^m \times \frac{1}{a^p},$$

i.e. $$a^{m-p} = a^m \times a^{-p}, \qquad \text{[}\textit{Def. IV.}$$

or $$a^{m+n} = a^m \times a^n.$$

CASE IV.—*Where both the indices are negative.*

Let $m = -p$, $n = -q$, where p and q are positive.

Then $$a^m \times a^n = \frac{1}{a^p} \times \frac{1}{a^q} \qquad \text{[}\textit{Def. IV.}$$
$$= \frac{1}{a^p \times a^q} = \frac{1}{a^{p+q}} \qquad \text{[Case I. or II.}$$
$$= a^{-(p+q)} \qquad \text{[}\textit{Def. IV.}$$
$$= a^{m+n}.$$

Note that the difficult cases are mostly where the indices have fractional values, and consequently introduce roots. In all these cases we raise the expressions to the corresponding powers and use the identity $(\sqrt[q]{x})^q = x$, which is the definition of $\sqrt[q]{x}$. See Case II. in §§ 76, 77, 79.

77. To prove that $(ab)^n = a^n b^n$ for all values of n.

CASE I.—*Where n is a positive integer.*

This has been proved in § 70.

CASE II.—*Where n is a positive fraction.*

Let $n = \dfrac{p}{q}$, where p and q are positive integers. Then $nq = p$.

Then
$$\{(ab)^n\}^q = \{\sqrt[q]{(ab)^p}\}^q$$
$$= (ab)^p \qquad \text{[\textit{Def. of Root}}$$
$$= a^p b^p. \qquad \text{[Case I.}$$

Also
$$(a^n b^n)^q = (a^n)^q (b^n)^q \qquad \text{[Case I.}$$
$$= (\sqrt[q]{a^p})^q . (\sqrt[q]{b^p})^q = a^p b^p.$$

Thus
$$\{(ab)^n\}^q = (a^n b^n)^q$$

whence
$$(ab)^n = a^n b^n.$$

CASE III.—*Where n is negative.*

Let $n = -p$, where p is positive.

Then
$$(ab)^n = \frac{1}{(ab)^p} \qquad \text{[\textit{Def. IV.}}$$
$$= \frac{1}{a^p b^p} \qquad \text{[Case I. or II.}$$
$$= \frac{1}{a^p} \cdot \frac{1}{b^p} = a^{-p} b^{-p} \qquad \text{[\textit{Def. IV.}}$$
$$= a^n b^n.$$

78. To prove that $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ for all values of n.

$$a = \frac{a}{b} \times b.$$

Hence
$$a^n = \left(\frac{a}{b} \times b\right)^n$$
$$= \left(\frac{a}{b}\right)^n \times b^n \qquad \text{[§ 77}$$
$$\therefore \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n.$$

79. To prove that $(a^m)^n = a^{mn}$ for all values of m and n.

CASE I.—*Where m and n are positive integers.*

This has already been proved in § 70.

CASE II.—*Where m and n are positive fractions.*

Let $m = \frac{p}{q}$, and $n = \frac{r}{s}$, where p, q, r, s are integers.

Then $$(a^m)^n = \left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = \text{(say) } k.$$

Now $$k^s = \left[\sqrt[s]{\left\{\left(a^{\frac{p}{q}}\right)^r\right\}}\right]^s$$

$$= \left(a^{\frac{p}{q}}\right)^r = a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \ldots \text{to } r \text{ factors.}$$

$$= a^{\frac{p}{q} + \frac{p}{q} + \ldots \text{to } r \text{ terms}} \qquad [\S\ 76$$

$$= a^{\frac{pr}{q}} = \sqrt[q]{a^{pr}}.$$

$$\therefore (k^s)^q = a^{pr}, \quad i.e. \quad k^{qs} = a^{pr} \qquad [\text{Case I.}$$

$$\therefore k = \sqrt[qs]{a^{pr}} = a^{\frac{pr}{qs}} = a^{mn}.$$

CASE III.—*Where m and n are both negative.*

Let $m = -p$ and $n = -q$, where p and q are positive.

Then $$(a^m)^n = \frac{1}{(a^m} \qquad [Def.\ IV.$$

$$= \frac{1}{\left(\frac{1}{a^p}\right)^q}$$

$$= \frac{1}{\frac{(1)^q}{(a^p)^q}} \qquad [\S\ 78$$

$$= \frac{1}{\frac{1}{a^{pq}}} \qquad [\text{Case I. or II.}$$

$$= a^{pq} = a^{mn}.$$

CASES IV. AND V.—*Where m is positive and n nejative, and vice versa.*

The proofs should now be obvious.

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